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RESEARCH ARTICLE

Foreword: Three-Valued Logics and their Applications

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1. Why a third truth value?

Three-valued logics belong to a family of nonclassical logics that started to flourish in the 1920s and 1930s, following the work of (Łukasiewicz, 1920), and earlier insights coming from Frege and Peirce (see (Frege, 1879), (Frege, 1892), (Fisch and Turquette, 1966)). All of them were moved by the idea that not all sentences need be True or False, but that some sentences can be indeterminate in truth value. In his pioneering paper, Łukasiewicz writes:

> "Three-valued logic is a system of non-Aristotelian logic, since it assumes that in addition to true and false propositions there also are propositions that are neither true nor false, and hence, that there exists a third logical value."

Łukasiewicz himself was urged to introduce a third truth value in order to model the notion of *possibility*, and to formalize Aristotle's insight that contingent sentences about the future can be indeterminate. Before him, Frege had identified two other reasons to think of sentences as neither true nor false. The first concerns sentences in which a proper name in a sentence *fails to refer* to an existing individual, as in "Odysseus was set ashore at Ithaca while sound asleep" (Frege, 1892). What Frege points out is that "anyone who seriously took the sentence to be true or false would ascribe to the name 'Odysseus' a reference". By contraposition, the failure of the name to have a reference should imply that the sentence fails to be either True or False. The second kind of case for which Frege thought sentences could be indeterminate concerns sentences involving *vague predicates*. For a predicate like "heap of beans", Frege points out that the classical induction principle cannot be applied, "on account of the indeterminateness of the notion "heap"" (Frege, 1879). Frege did not propose a three-valued logic in relation to those observations, but his remarks find an echo in the supervaluationist system proposed by (van Fraassen, 1966) to account for the semantics of nonreferential singular terms, and used a decade later by others to account for vagueness (Fine (1975), Kamp (1976)).

Besides contingency, reference failure, and vagueness, the development of three-valued logics, both early and recent, can be associated to at least three other phenomena of interest, in which the notion of indeterminacy plays a central role, namely conditionals, computability, and the semantic paradoxes. In 1935, De Finetti proposed a three-valued treatment of indicative conditional sentences in relation to probability, intended to model cases in which the antecedent of the conditional is false, leaving the conditional undefined (De Finetti, 1936). In De Finetti's table, the conditional is true when antecedent and consequent are both true, and false when the antecedent is true and the consequent is false; but his main intuition is that a conditional should be undefined in all other

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cases, especially when the antecedent turns out false. In 1937, Bochvar published an article in which he investigated a three-valued calculus which he applied to the analysis of Russell's paradox and Grelling's paradox to establish certain paradoxical sentences as "meaningless" (Bochvar, 1937). Unlike De Finetti's, Bochvar's logic coincides with classical logic when all arguments are defined, but it assigns the value undefined to a sentence whenever one of its components is undefined. A year later in 1938, Kleene proposed three-valued tables for the logical connectives in relation to his theory of partial recursive functions, to represent cases in which the truth of a sentence might not be decided by means of Turing machine (Kleene, 1938). On his account, a connective takes a classical value when all ways of completing the assignment of the undefined value by a classical value converge to the same value; in all other cases, the function stays undefined (see (Kleene, 1952), and George this issue for details).[1]

Quite remarkably, all of those areas of applications have remained active fields of research until today. Also, various of those frameworks, originally developed to approach one class of phenomena, have been transposed to tackle other phenomena. A representative case is Kleene's (strong) logic, originally developed to account for partiality in computation, which has been applied to the treatment of presupposition projection (see (Beaver, 2001) for an overview, and George this issue), to the semantic paradoxes (see (Kripke, 1975), (Field, 2008)), and to vagueness (viz. (Körner, 1955), (Tye, 1994), (Cobreros et al., 2012b)). A second case is the supervaluationist framework, originally developed by (van Fraassen, 1966) to account for nonreferential singular terms, but prefigured in (Mehlberg, 1958)'s analysis of vagueness and further elaborated to deal with presupposition, self-referential truth and vagueness (see (van Fraassen, 1968), (Kripke, 1975), (Fine, 1975), (Kamp, 1976)). A third case is Bochvar's logic, rediscovered independently by (Halldén, 1949) in relation to vagueness (see Williamson (1994)), and by (Kleene, 1952) in relation to the theory of computation, and later discussed in relation to presupposition (see (Beaver, 2001)).

2. What does the third truth value stand for?

2.1 *Third truth value, or lack of truth value?*

Unlike Łukasiewicz, Frege did not think of indeterminacy as a separate truth value, but rather as the lack of a truth value. This difference was later to be reflected by a difference between two kinds of three-valued systems, with on the one hand *three-valued logics* properly so-called, such as Łukasiewicz's, in which the third truth value is admitted on a par with the values True and False; and on the other hand, systems of *partial two-valued* logics based on truth-value gaps, such as supervaluationist logics (van Fraassen, 1966), (Kamp and Partee, 1995), in which a sentence fails to receive a value True or False.

In the supervaluationist framework, a sentence is either semantically defined or undefined. When undefined, one considers all possible ways of assigning it a classical truth-value. It is then called (super)-true if it takes the same classical value True under all ways of making it defined, (super)-false if it takes the same value False under all ways of making it defined, and it is neither true nor false otherwise (van Fraassen, 1966). For example, if John is a borderline case of a bald man, "John is bald" will be undefined to begin with, and "John is bald" fails to be either super-true or super-false, since the sentence can be true or false. On the other hand, "John is bald or John is not bald" will

[1] This is the so-called strong Kleene scheme; the so-called weak Kleene scheme, also discussed in (Kleene, 1952), is in fact equivalent to Bochvar's.

be super-true, since the disjunction is classically True under all classical assignments of a value to "John is bald" (viz. (Mehlberg, 1958), (Fine, 1975)).

The situation is very different in Łukasiewicz original system. Łukasiewicz symbolizes True by 1, False by 0, and the third value $\frac{1}{2}$ stands for "possible".[1] In his system, disjunction patterns truth-functionally and corresponds to the maximum of the value of each disjunct, and similarly the value of the negation is 1 minus the value of the negated sentence. This implies that the law of excluded middle ($A \vee \neg A$) fails to take the value 1 under all assignments (when A gets the value $\frac{1}{2}$).

Because of that, one may think that any trivalent truth-functional system is one in which the third truth value needs to stand as a primitive notion. But it is not so. For example, one can easily super-impose a truth-functional architecture for the connectives on a supervaluationist understanding of the values 1, 0 and $\frac{1}{2}$ as assigned to atomic sentences, taking those values to be defined rather than primitive (see (Ripley, 2012b) for an illustration of this approach: a truth-value is basically the set of classical values a sentence can take, namely $\{1\}$ when "supertrue", $\{0\}$ when "superfalse", and $\{1, 0\}$ otherwise; see also (Cobreros and Tranchini, 2014) for a recent discussion of value-functionality in supervaluationism). A related perspective on 3-valued logics is to view the three values as only a subset of values within a 4-valued architecture (see (Dunn, 1969), (Belnap, 1977)), where the values True, False, Both and Neither can be seen as resulting from a relational rather than functional 2-valued semantics (viz. True is assigned to a sentence that is related to 1 only, False to one that is related to 0 only, Both to a sentence that is related to both 1 and 0, and Neither to a sentence with no relata, see (Beall and van Fraassen, 2003) for details).

2.2 *Ontic vs. epistemic interpretations*

Quite generally, the introduction of a third truth-value in logic raises delicate questions of interpretation (see (Haack, 1996) for an overview). In many cases, the interpretation of the third truth-value oscillates between an *ontic* interpretation and an *epistemic* interpretation (also called *informational*, in particular in relation to the bilattice interpretation of 3-valued logic within a 4-valued framework, see Belnap (1977), Fitting (1991) and Martinez, this issue).

In Łukasiewicz's original system, for example, "possible" appears to be taken in the sense of "factually unsettled". De Finetti, by contrast, asserts that for him propositions can only be true or false, but that he takes the third truth value to represent subjective uncertainty about the proposition (De Finetti, 1936). (Kleene, 1952) lets the third truth value stand for "undecidable by the algorithms whether true or false", also favoring an epistemic interpretation. In both cases, however, the relevant sense of "epistemic" needs to be qualified, since for both of them instances of excluded middle ($A \vee \neg A$) remain undefined rather than true. Several systems of three-valued logic, as it turns out, are susceptible of both interpretations. In Priest's Logic of Paradox (Priest, 1979), where the third truth value stands for "both true and false", the latter is interpreted ontically. But informational interpretations of the same system have been proposed (see (Belnap, 1977), where the "both" value is interpreted as "*being told* both falsity and truth", and (Lewis, 1982) for discussion).

A more neutral stance on the interpretation of the third truth value is the following: depending on the application, some sentences are assigned a special semantic status, *other* than True or False, to reflect the fact that such sentences are not assertible in the

[1]In Łukasiewicz's original notation, the third value is represented as 2; see for example Martinez, this issue, for the use of the latter convention.

way True and False sentences are, and that they do not necessarily support the same inferences. For instance, in theories of presupposition projection (see (Peters, 1979), (Beaver and Krahmer, 2001); George, this issue), the third truth value is used to represent cases in which a sentence is judged infelicitous, for cases of presuppositional failure. The value $\frac{1}{2}$, in this case, stands for "infelicitous", or "defective", and the choice of a particular 3-valued scheme for the connective is dictated by the problem of determining how the defectiveness of a constituent sentence percolates up to large sentences in which it is embedded (see George, this issue, for an exposition). Similarly, in theories of vagueness, the value $\frac{1}{2}$ is used to assign a special semantic status to borderline cases (see (Avron and Konikowska, 2008), (Cobreros et al., 2012b)), and again the choice of a particular valuation scheme will depend on how one thinks the vagueness of a subsentence is inherited to larger sentences (see (Ripley, 2012b), (Alxatib and Pelletier, 2011), (Cobreros et al., 2012b), and (Alxatib et al., 2013) for recent discussions about this point). Likewise, if we consider Bochvar's treatment of paradoxical sentences, the third value is meant to stand for "meaningless", to separate a class of sentences from those that are susceptible of True vs. False evaluation. The same concern is at stake in Kripke's theory of truth, where Kleene's 3-valued logic is used to delineate sentences such as the Liar or the Truth-Teller, which Kripke characterizes as "ungrounded" (see (Kripke, 1975)).

3. Trivalent connectives and logical consequence

From a logical point of view, 3-valued logics can be seen as only a first step within the broader family of many-valued logics. Three-valued logics are sometimes viewed narrowly as offering only a limited surplus of freedom over two-valued logics for that matter, mostly by enlarging the space of interpretations for the logical connectives (viz. the choice of 3^9 truth-functional tables for a binary connective, vs. 2^4 in a 2-valued setting). This is already a significant increase of possibilities (see George, this issue for discussion), especially when it comes to modeling elusive connectives like the conditional (see (Cantwell et al., 2008), (Baratgin et al., 2012) for recent discussions in that area). But depending on the applications it is sometimes deemed more desirable to introduce even more truth-values (viz. (Smith, 2008) for such a view in the case of vagueness).

Upon closer examination, 3-valued logics offer more options. First of all, as already mentioned, the interpretation of connectives in a 3-valued setting need not be truth-functional: the third value can be assigned non-truthfunctionally (as in supervaluationism), or nondeterministically (by allowing a allowing a set, rather than a unique value in some cases, see (Avron and Konikowska, 2008), (Avron and Zamansky, 2011)). Secondly, the definition of logical consequence in a 3-valued setting leaves various choices open. When logical consequence is interpreted in terms of the preservation of designated values, one basic choice is between the preservation of the value $\{1\}$ (preservation of Truth, or strong consequence), and the preservation of non-zero values $\{1, \frac{1}{2}\}$ (Preservation of non-Falsity, or weak consequence). A typical illustration of this choice is given in the duality between paracomplete systems (such as Kleene's strong logic K3, or supervaluationism, in which $B \nvDash A, \neg A$) and paraconsistent systems (like Priest's logic LP, dual to K3, and subvaluationism, dual to supervaluationism, in which $A, \neg A \nvDash B$).[1]

[1]For more on paracomplete vs. paraconsistent logics, see Avron, this issue. On subvaluationism, see (Hyde, 1997) and (Cobreros, 2013). Unlike K3 and LP, super- and sub-valuationism correspond to *weakly* paracomplete and paraconsistent systems respectively, in which the law of excluded middle $B \vDash A \vee \neg A$ and the conjunctive version of the explosion principle $A \wedge \neg A \vDash B$ continue to hold respectively. See (Ripley, 2012b) and (Cobreros et al., 2012a) for a comparison between those systems in relation to vagueness.

Further combinations of those choices have also been considered: in the case of 3-valued conditional logics, for example, it is standard to require both directions of preservation simultaneously, which corresponds to intersecting the logics (see (McDermott, 1996) in the case of conditionals; see (Field, 2008), chapter 3, Appendix on Kleene logics and other DeMorgan logics). Another possibility is to define consequence in a mixed way: when all premises take value 1, no conclusion should take value 0; or dually, when no premise takes the value 0, some conclusion should take the value 1 (see (Frankowski, 2004) and (Malinowski, 1990), who call those relations p-consequence and q-consequence respectively, and (Nait-Abdallah, 1995) and (Bennett, 1998) for related accounts). This view of logical consequence has been advocated and taken further in recent years in relation to the treatment of the paradoxes of vagueness, as well as the semantic paradoxes (see Zardini (2008) on this view in a many-valued setting, and (van Rooij, 2011), (Cobreros et al., 2012b), (Cobreros et al., 2012a), (Ripley, 2012a), and (Cobreros et al., 2014) for corresponding explorations in a 4-valued and 3-valued setting specifically). These developments have opened new logical perspectives, in particular regarding the interest of nontransitive logics toward a unified treatment of the paradoxes of vagueness and of self-referential truth.

On the technical side, further recent explorations of 3-valued systems have concerned on the one hand the proof theory of 3-valued logics. The paradigmatic logics of Łukasiewicz, Kleene and De Finetti all share a common core, which concerns the interpretation of negation, disjunction and conjunction, but they differ in systematic ways on the interpretation of the conditional (De Finetti's is undefined whenever the antecedent is not true or the consequent is undefined; Kleene's is handled as equivalent to $\neg A \vee B$; and Łukasiewicz's differs from Kleene's in that the conditional gets the value 1 when both antecedent and consequent are undefined). The number of variants of those systems is too large to review here, but the relation between these logics and the search of integrated proof systems for them has been an object of continued interest (see particularly (Avron and Konikowska, 2008) for an investigation along those lines in the case of the Łukasiewicz and Kleene logics).

Another line of recent interest has concerned the relation between 3-valued logics and modal logics. It is known since the early days of many-valued logics that normal modal operators are not adequately expressible by means of n-valued truth-functional connectives (see (Dugundji, 1940)). This gives a natural sense in which modal logics are more expressive than n-valued logics (see (Prior, 1953) for related arguments against the adequacy of Łukasiewicz's trivalent approach in the temporal case). A natural question is whether many-valued systems, and in particular 3-valued systems, can be embedded in a 2-valued modal system. A positive answer to this question has been given in (Kooi and Tamminga, 2013), yielding further insights regarding the interpretation of the third truth-value (see also (Cobreros et al., 2012b) for a related perspective).

4. Overview of the issue

The papers collected in this special issue of JANCL originate from a workshop on Trivalent Logics and its Applications organized by Paul Égré and David Ripley at the ESSLLI 2012 Summer School in Opole. Several of the papers presented on that occasion appear in this volume, with the inclusion of additional papers submitted independently (all of the papers were double-blind-reviewed, and the papers submitted at ESSLLI 2012 received a first preliminary round of blind reviews on the occasion of the workshop). The papers cover a wide range of topics, and together they overlap with all of the areas of applications we listed above.

4.1 *Conditionals*

Four of the papers included in this issue deal with the application of 3-valued logics to the treatment of conditional sentences in natural language.

Daniel Rothschild, in his paper, examines the usefulness of a 3-valued semantics for indicative conditionals in relation to the problem of the assignment of a probability value to such sentences. Since the so-called triviality results of Lewis 1976, it is known that there is no obvious way of assigning probabilities to conditional sentences in accordance with the idea that the probability is the conditional probability of the consequent, given the antecedent. Rothschild discusses a way, building on De Finetti's original table for the conditional (rediscovered independently by (Belnap, 1970)), to achieve this result without triviality. *Mathieu Vidal* examines 3-valued logics for conditionals in the light of an alleged oddity of the 2-valued treatment of the conditional as a material conditional, namely the classically valid inference from $A \wedge B \supset C$ to $(A \supset C) \vee (B \supset C)$. Vidal argues that this inference is at odds with ordinary reasoning in mathematics, and shows that a family of 3-valued conditionals, including De Finetti's, falls prey to the same objection.

The next two papers explore the connection between conditionals, 3-valued logics and logic programs (see in particular (Fitting, 1985) and (Fitting, 1991) for an overview of the connection of logic programming with Kleene's strong logic). *Emmanuelle Dietz, Steffen Hölldobler and Christoph Wernhard* in their paper start out from the consideration of psychological data regarding the conditional, namely R. Byrne's suppression task, indicating that classical 2-valued logics fails to account for the nonmonotonicity of the conditional in ordinary reasoning. They follow the inspiration of (Stenning and Van Lambalgen, 2008) to give an account of the suppression task in the framework of logic programming, but based on an adaptation of Łukasiewicz 3-valued logics instead of Kleene's K3. In her paper, *Katrin Schulz* too considers an extension of Stenning and van Lambalgen's approach to deal with nonmonotonicity and with the semantics of counterfactual conditionals more generally. The adaptation she proposes permits to represent the dependence of judgments involving conditionals relative both to facts and to general rules. In her approach, logical programs represent only general laws, and facts are represented as the set of literals made true or false within a trivalent ground model. Schulz's main contention is that an account of conditionals based on logic programming is more predictive and more tractable than an account based on standard possible world semantics.

4.2 *Presupposition, Truth and Vagueness*

The next group of papers concerns applications of three-valued logics to the treatment of presupposition, self-referential truth, and vagueness respectively.

In his paper, *Benjamin George* deals with the debated problem of presupposition projection. The problem of presupposition projection is the problem of deriving the presuppositions of a complex sentence from the presuppositions of its parts. Various systems have been proposed since the 1970s, including three-valued approaches (Peters, 1979), dynamic semantics ((Heim, 1983), (Beaver, 2001)), and more recently pragmatic accounts based on a two-valued approach (see (Schlenker, 2007)). George in his paper offers to vindicate the trivalent account of (Peters, 1979). Peters's scheme for the connectives is essentially an asymmetric version of Kleene's strong logic, intended to reflect a corresponding asymmetry regarding presupposition projection (viz. "the King of France is bald, and France has a king", vs. "France has a King and there is a King of France"). George states the conditions under which one can obtain a correspondence between the dynamic account of Heim-Beaver and Peters' static 3-valued account; he also shows the sense in which the semantic rules attached to Peters' connective are nonarbitrary, based on Kleene's informational justification for his scheme; finally, he proves several formal results regarding the extension of Peters' system to generalized quantifiers.

José Martinez in his paper deals with the conditions under which a propositional language admitting self-reference can have a truth predicate satisfying the identity of truth, namely a predicate T such that $T(\overline{\phi})$ is identical in truth value with ϕ for any sentence ϕ. The Liar paradox shows that truth predicates in that sense are not definable in a 2-valued setting, but since the work of Kripke and others on truth, it is known that various trivalent schemes are compatible with the existence of a truth predicate. From an algebraic point of view, the definability of a truth-predicate can be viewed as equivalent to the existence of a fixed point for a particular operator (Kripke's so-called *jump operator*, which assigns a positive and negative extension to the truth predicate at each ordinal; see also (Fitting, 1985) on the relation between monotone operators in logic programming and Kripke's theory of truth). The problem investigated in Martinez's paper consists, given a set of truth values, to characterize the class of propositional schemes and functions which have that fixed point property. Martinez proves several results in relation to this problem, not restricted to the 3-valued case, but holding of finite many-valued logics quite generally. A particular application of those results is provided regarding the admissibility of various unary operators in 3-valued and 4-valued logics, whose intent is to express different kinds of negation. In particular, Martinez shows that the weak Kleene scheme affords more expressiveness than the strong Kleene scheme to state facts such as "this sentence is undetermined/meaningless".

Heather Burnett's paper concerns the semantics of vague gradable adjectives (such as "rich", "empty", "safe"). One of the hallmarks of vague predicates is their relative insensitivity to small changes (aka the tolerance principle), and their susceptibility to sorites paradox (viz. Smith (2008)). Following (van Rooij, 2011) and earlier work by (Zardini, 2008), (Cobreros et al., 2012b) proposed a semantics for vague predicates, the strict-tolerant account, which validates the tolerance of vague predicates without paradox, and which basically combines features of the strong Kleene logic with its dual, Priest's LP. Burnett's paper extends the strict-tolerant framework of Cobreros et al. (originally set up in a two-valued logic augmented with similarity relations, but equivalent in a deep sense to a three-valued logic) to account for the relation between the positive form and the comparative form of gradable adjectives. Specifically, she accounts for three classes of inferences relating the positive and the comparative form – which she calls scalarity, maximality and evaluativity – and she uses them to give a model-theoretic characterization of three subclasses of gradable adjectives, namely total, partial and relative adjectives. To achieve this result, her system combines the TCS framework of (Cobreros et al., 2012b) with a delineation semantics inspired from (Klein, 1980). Besides extending the notions of strict and tolerant interpretations to comparative relations, her system partializes both notions of interpretation with an additional third value, to model the relativity of such interpretations to a comparison class argument.

4.3 *Proof theory and connections to other logics*

The last group of papers in this issue consists of three more technical papers. Two of those look at the relevance of 3-valued logics and 4-valued logics for the characterization of validities in other systems. *Stepan Kuznetsov* in his paper considers two versions of the Lambek calculus, the Lambek calculus with the unit constant and the Lambek calculus with the empty set constant, and shows how to get soundness results for two fragments of those, respectively based on the three-valued calculi RM3 and K3. *Tin Perkov* presents a semantics combining 4-valued logics with a Kripke-style semantics for intuitionistic logic, allowing him to characterize both classical validities and intuitionistic validities in terms of a difference between weak vs. strong validities (sentences taking the value $\{1, \frac{1}{2}\}$ vs. sentences taking the value $\{1\}$). The characterization is given both for the propositional and the first-order fragment of each calculus. In his paper, finally, *Arnon Avron* focuses on the duality between paracomplete and paraconsistent logics from a proof-theoretic perspective. Avron defines a family of Gentzen-type systems which he calls quasi-canonical, and in which the rule for negation is restricted to produce either a paracomplete or a paraconsistent logic (but not both). Avron shows that both kinds of systems can be constructively characterized in terms of a coherence criterion relating premises and conclusions in a sequent. He uses the framework of nondeterministic 3-valued matrices to produce soundness and completeness results for the associated logics, and to establish when a paraconsistent or paracomplete system admits cut-elimination.

5. Perspectives

The ambition of this special issue and of the workshop that has preceded it is to enhance the value of 3-valued logics for a range of applications, and to make bridges between the various areas of philosophy, logic and linguistics in which such systems were launched and given their initial motivation. We believe the papers brought together in this issue to provide a representative sample of applications relative to the scope of 3-valued logics, even though several topics could not be included here. On the technical side, one aspect that is not covered concerns the definition of proof systems for weakly paraconsistent/paracomplete logics or their combination (viz. sub- and super-valuationist systems, see (Cobreros et al., 2012a) for an overview, and (Tranchini, 2013) for recent work on proof theory). On the conceptual side, one difficult issue that is only partly discussed here is whether the various phenomena we listed at the outset, in particular presupposition projection, vagueness, and the semantic paradoxes, are susceptible of a unified treatment within an integrated system, or whether a pluralistic conception is more adequate (see (Cobreros et al., 2014), (Zehr, 2014) and (Spector, 2012) for recent explorations of those topics). We also refer the interested readers to the excellent special issues of *Studia Logica* on Truth Values (Shramko and Wansing, 2009) and Many-Valued Logic and Cognition (Ju and Mundici, 2008) for an overview of further work in trivalent logics.

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