Introduction to Mathematical Logic

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Outline of the course

1. Preliminaries
2. Propositional Logic
3. Tree method for PL
4. First-order Predicate Logic
5. Tree Method for FOL
6. Expressiveness of FOL

1. Preliminaries

What is Logic?

- A theory of valid inferences, or of what follows from what

John is a professor or a criminal

John is not a professor

John is a criminal

  Every professor is a man

  No singer is a man

  No singer is a professor

Every professor is a man

No singer is a professor

No singer is a man

The notion of logical consequence

- An inference/argument consists of a set of premises and a conclusion
- An argument is said to be deductively valid when the conclusion follows logically from the premises

Problem: how to define this notion of logical consequence?

Motto: The conclusion can't be false if the premises are supposed to be true.

Logic and Language

The validity of an argument depends on the use of certain logical expressions.

John is a professor or a criminal

John is not a professor

John is a criminal

Fanny is a singer or a painter

Fanny is not a singer

Fanny is a painter

Formal logic

A theory of valid inferences for a language whose syntax can be rigorously defined.

• "A logic is a language equipped with rules for deducing the truth of one sentence from that of another. Unlike natural languages such as English, Finnish, and Cantonese, a logic is an artificial language having a precisely defined syntax" (S. Hedman, 2004)

• "I would say that logic is the systematic study of logical truths. Pressed further, I would say that a sentence is logically true if all sentences with its grammatical structure are true. Pressed further still..." (W. V. O. Quine, 1970).

Example : Aristotle's syllogistic

No man walks

Some man walks

Every man walks

Not every man walks

Every sentence of the fragment is of the form: QAB, where Q is one of the four quantifiers, and A and B are predicates.

A syllogism: an inference with two premises and a conclusion, each of the form Q A B

ex : Every man is mortal, no god is mortal ; no man is a god.

The four figures

	1st fig.	2nd fig	3rd fig	4th fig.
Major Premise	Q **B**C	Q C**B**	Q **B**C	Q C**B**
Minor Premise	Q A**B**	Q A**B**	Q **B**A	Q **B**A
Conclusion	Q AC	Q AC	Q AC	Q AC

There are $4^4 = 256$ possible inference schemata (4 figures $\times$ 4 possible quantifiers Q in each premise and conclusion)

How many of them are valid? Only 24 (assuming A, B and C each contain at least one individual).

Expressive limitations

Some valid inferences cannot be represented in aristotelian logic:

No student knows every professor

Some student knows every assistant

Some professor is not an assistant

If every man is wise, then every banker is honest

Some banker is not honest

Not every man is wise

Logic and Grammar

Frege (*Begriffschrift*, 1879): "so far logic has always been tied too closely to language and grammar"

- Aristotle does not handle quantified expressions in object position
- No proper treatment of transitive verbs (like *know*)
- No proper treatment of sentential connectives, like conditionals (*if...then*"), conjunction (*and*), disjunction (*or*).
- No proper treatment of unrestricted quantification : *someone*
- No proper treatment of singular terms: *Socrates*

What we shall study

- propositional logic : a logic of sentences, in which inferences rest on the use of logical words like *not*, *if*, *and*, ...
- first-order predicate logic : an enrichment of propositional logic, originally defined by Frege, and more expressive than Aristotelian logic
- We will see, however, that still some sentences of natural language cannot be expressed in FOL.

Formal logic

- a language with a precise syntax
- a way of assigning meanings to logical expressions (semantics)
- a way of computing logical relations between expressions (proof theory, model theory)

Bibliography

The following books are very useful and have been used for the preparation of this course:

- Shawn HEDMAN (2004) *A First Course in Logic*, Oxford Texts in Logic. [chap. 1 and 2]
- Dirk van DALEN (2004) *Logic and Structure*, Springer, 4th ed.
- Bernard RUYER (1990) *Logique*, PUF.
- Raymond SMULLYAN(1968), *First-Order Logic*, Dover.

Some mathematical tools

We need a few mathematical tools, not much...

• Sets, membership and inclusion : $x \in X$ (x is a member of X)

$X \subseteq Y$: every member of X is a member of Y

ex : $0 \in \{0, 3\}$, $\{0\} \subseteq \{0, 3\}$

• Relations : a binary relation is a set of ordered pairs. For instance, if Jack is father of Mary, and John father of Jill, the relation "father" can be described by:

$R = \{(Jack, Mary), (John, Jill)\}$

$R' = \{(Mary, Jack), (Jill, John)\}$ is a different relation ("daughter of")

2. Propositional logic

Propositional Logic

Propositional logic is also called sentential logic : atomic formulas stand for full declarative sentences, irrespective of their inner structure.

John is smoking	$\rightsquigarrow$	p
Mary has two brothers	$\rightsquigarrow$	q

It is not the case that John is smoking : $\neg p$

John is smoking or Mary has two brothers : $(p \vee q)$

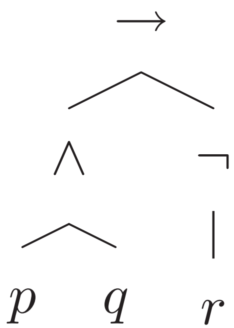
John is smoking and Mary has two brothers : $(p \wedge q)$

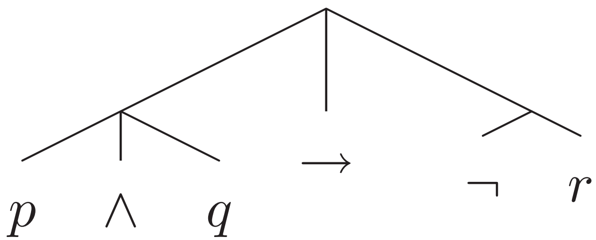
If John is smoking then Mary has two brothers : $(p \rightarrow q)$

Syntax of propositional logic

- A set At of atomic formulas : $p, q, r, p', q', r', ...$
- If F and G are formulas, so are: $\neg F$, $(F \wedge G)$, $(F \vee G)$, $(F \rightarrow G)$.
- Nothing else is a formula.

ex: $((p \wedge q) \rightarrow \neg r)$





Abbreviations

• Practically, we can drop a few brackets if we give the connectives some priority :

- $\neg$ has the smallest possible scope

- $\wedge$ and $\vee$ have smaller scope than $\rightarrow$ or $\leftrightarrow$

- $((F \vee G) \vee H)$ to $(F \vee G \vee H)$, and same with $\wedge$

- Leave the outermost brackets when there is no ambiguity

ex:

• $p \wedge q \rightarrow \neg r$ stands for $((p \wedge q) \rightarrow \neg r)$, distinct from $p \wedge (q \rightarrow \neg r)$

• $\neg r \wedge p$ stands for $(\neg r \wedge p)$, distinct from $\neg(r \wedge p)$.

Semantic interpretation

Formulas are interpreted by truth-falsity assignments, and by the rules for interpreting the connectives.

An assignment is a function $s : At \to \{0, 1\}$

F	$\neg F$
1	0
0	1

F	G	$(F \wedge G)$	$(F \vee G)$
1	1	1	1
1	0	0	1
0	1	0	1
0	0	0	0

The material conditional

F	G	$(F \to G)$
1	1	1
1	0	0
0	1	1
0	0	1

- If the antecedent is true and the consequent false : false
- If antecedent true and consequent true : true
- What if the antecedent is false?

ex : "if it's raining, then there are mushrooms". What if uttered when it's not raining?

Truth tables

• **Compositionality principle:** given any complex formula F of PL, it is possible to compute the truth value of that formula from the truth-value of its atomic components :

ex : $F = ((p \wedge q) \rightarrow \neg r)$

	p	q	r	$(p \wedge q)$	$\neg r$	$((p \wedge q) \rightarrow \neg r)$
s	1	0	1	0	0	1

• Given an assignment s, and a complex formula F, we shall write $s(F)$ to denote the value of F induced by s.

Validity, Satisfiability

Definition : A formula F is **valid** if, for every TF-assignment s to its atomic formulas, $s(F) = 1$

Definition : A formula is **satisfiable** if, for there is an assignment s such that $s(F) = 1$.

Definition : A formula is **unsatisfiable** if for every assignment s, $s(F) = 0$.

Theorem : a formula F is valid iff $\neg F$ is not satisfiable

Proof : suppose F valid : for every s, $s(F) = 1$, so $s(\neg F) = 0$, and $\neg F$ is unsatisfiable. Conversely, if F is not satisfiable, then there for every s, $s(\neg F) = 0$, ie $s(F) = 1$, and F is valid.

Three kinds of formulae

- A valid formula F is called a **tautology**

Notation : $\models F$

ex : $(p \vee \neg p)$

- An unsatisfiable formula F is called a **contradiction**

ex : $(p \wedge \neg p)$

- Some formulae are satisfiable but not valid (**neutral** formulae)

ex : $(p \wedge q)$

unsatisfiable	neutral	valid

The method of truth-tables

Decision problem for PC: Given a formula of PC, how to determine whether it is satisfiable? a validity? a contradiction?

Answer : truth-tables give us an algorithm

1) Write the truth-table with all relevant assignments for F

2) Compute the value of F under each assignment

3) - If the column for F contains 1 everywhere : valid

- If the column for F contains 0 everywhere : contradiction

- If the column for F contains at least a 1 : satisfiable

Theorem : Propositional logic is decidable

Proof : suppose F contains n distinct atoms. Then the truth-table for F has at most 2^n rows. The computation stops after finitely many steps.

Show that $\neg p \rightarrow (p \rightarrow q)$ is a tautology.

p	q	$(\neg p \quad \rightarrow \quad (p \rightarrow q))$
1	1	
1	0	
0	1	
0	0	

p	q	$(\neg p \quad \rightarrow \quad (p \rightarrow q))$
1	1	0
1	0	0
0	1	1
0	0	1

p	q	$(\neg p$	$\rightarrow$	$(p \rightarrow q))$
1	1	0		1
1	0	0		0
0	1	1		1
0	0	1		1

p	q	$(\neg p$	$\rightarrow$	$(p \rightarrow q))$
1	1	0	1	1
1	0	0	1	0
0	1	1	1	1
0	0	1	1	1

Logical consequence

Def: a formula F is a logical consequence of a set of formulae Γ if every assignment that gives the value true to all formulae of Γ yields the value true to F.

In other words : it is impossible for the formulae of Γ to be all true without F being true.

Notation : $\Gamma \models F$

example : $\Gamma = \{(p \wedge q), (q \rightarrow \neg r)\}$, $F = \neg r$

Clearly : $(p \wedge q) \models q$

Moreover : $q, (q \rightarrow \neg r) \models \neg r$

So : $(p \wedge q), (q \rightarrow \neg r) \models \neg r$

Some properties of logical consequence

- $F \models F$ (reflexivity)
- If $\Gamma \subseteq \Gamma'$, then if $\Gamma \models F$, $\Gamma' \models F$ (monotonicity)
- If $F \models G$ and $G \models H$, then $F \models H$ (transitivity)

Consequence and implication

Deduction Theorem : $\Gamma, F \models G$ iff $\Gamma \models (F \rightarrow G)$

Proof ($\Rightarrow$, converse is similar): suppose $\Gamma, F \models G$; for every s that satisfies both F and all formulae in Γ, s satifies G. Take an s that satisfies all formulae in Γ. Suppose it does not satisfy $(F \rightarrow G)$: then it should satisfy F and not G. But if it satisfies F and Γ, it ought to satisfy G : contradiction.

• This gives us a way of checking whether a formula F is a logical consequence of a set Γ of formulas (when Γ is finite) : check whether $\models (\bigwedge \Gamma \rightarrow F)$ by truth-tables ($\bigwedge \Gamma$ denotes the conjunction of all formulas of Γ)

Logical Equivalence

Two formulas are logically equivalent iff they are satisfied by the same assignments, or iff each one is a logical consequence of the other. Notation : $F \equiv G$

Ex : De Morgan's rules :

$\neg(F \wedge G) \equiv (\neg F \vee \neg G)$

$\neg(F \vee G) \equiv (\neg F \wedge \neg G)$

The biconditionnal

F	G	$(F \leftrightarrow G)$
1	1	1
1	0	0
0	1	0
0	0	1

- $(F \leftrightarrow G) \equiv ((F \rightarrow G) \wedge (G \rightarrow F))$
- Property : $F \equiv G$ iff $\models (F \leftrightarrow G)$

Useful equivalences

$(F \to G) \equiv (\neg F \vee G)$ (conditional as disjunction)

$((F \wedge G) \wedge H) \equiv (H \wedge (G \wedge H))$ (associativity of "and")

$((F \vee G) \vee H) \equiv (H \vee (G \vee H))$ (assoc. of "or")

$\neg\neg F \equiv F$ (double negation)

$(F \wedge (G \vee H)) \equiv ((F \wedge G) \vee (F \wedge H))$ (distrib. of "and" over "or")

$(F \vee (G \wedge H)) \equiv ((F \vee G) \wedge (F \vee H))$ (distrib. of "or" over "and")

Exercices

1. Translate the following sentences in propositional logic:

a. If John comes to the party, then Mary will not be happy.

b. Unless John comes to the party, Mary will not be happy.

c. If Bill jumps and Mary doesn't make a leap, Sam will have to do a gigantic step.

2. Compute whether the following formulae are tautologies or not:

$(p \to q) \to (\neg q \to \neg p)$

$(p \to q) \to (q \to p)$

$(((p \to q) \to p) \to p)$

$(p \to (q \to p))$

3. (For those who are familiar with sets). Let S be the set of all assignments $s : \{p, q, r, ...\} \rightarrow \{0, 1\}$. If F is a formula, we define $[\![F]\!]$ (the proposition expressed by F) as $[\![F]\!] = \{s \in S; s(F) = 1\}$.

a) Show:
$$\begin{aligned} [\![(F \wedge G)]\!] &= [\![F]\!] \cap [\![G]\!] \\ [\![(F \vee G)]\!] &= [\![F]\!] \cup [\![G]\!] \\ [\![\neg F]\!] &= S - [\![F]\!] \end{aligned}$$

b) Show that:

- $\models F$ iff $[\![F]\!] = S$
- F is satisfiable iff $[\![F]\!] \neq \emptyset$
- $\models F \rightarrow G$ iff $[\![F]\!] \subseteq [\![G]\!]$
- $\models F \leftrightarrow G$ iff $[\![F]\!] = [\![G]\!]$

Normal forms

Def. A formula is in disjunctive normal form (DNF) iff it is a disjunction of conjunction of atoms or negated atoms.

ex : $(p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg r)$

Thm. Every formula F of PL is equivalent to a formula F' in DNF

Proof (sketch) : (i) build the truth-table for F ; (ii) translate each 1-row into a conjunction of atoms (for value 1) or negated atoms (for value 0); (iii) take F'=the disjunction of all such conjunctions.

Clearly : F' is in DNF. By construction, if $s(F) = 1$, then $s(F') = 1$; conversely, if $s(F') = 1$, then one of the disjuncts of F' is satisfied by s, but again by construction, $s(F) = 1$ at the corresponding row.

Example

p	q	$((p \to q) \to \neg q)$		
1	1	0		
1	0	1	$\rightsquigarrow$	$(p \wedge \neg q)$
0	1	0		
0	0	1	$\rightsquigarrow$	$(\neg p \wedge \neg q)$
				$(p \wedge \neg q) \vee (\neg p \wedge \neg q)$

Functional completeness

• Corollary: every formula of PL is equivalent to a formula written only with the symbols $\neg$ and $\wedge$.

Proof: use the fact that $(F \vee G) \equiv \neg(\neg F \wedge \neg G)$.

• Thm: Let f be a primitive n-ary truth-functional connective defined by its truth-table over the atoms $p_1, ..., p_n$. There is a formula F, written only with $\neg$ and $\wedge$ and the p_i and such that $\models f(p_1, ..., p_n) \leftrightarrow F$.

Proof: use the DNF method again, and again the definition of $\vee$ from $\neg$ and $\wedge$.

Exercises

1. Put the following formula in Disjunctive Normal Form:

$(p \leftrightarrow q) \wedge (\neg r \rightarrow p)$

2. Every formula can be written in Conjunctive Normal Form (CNF), ie as a conjunction of disjunction of atoms or negated atoms. Try to prove this fact, by analogy to what we did with DNF.

3. The Sheffer stroke ('*nand*') is a binary connective defined as $(F|G)$ is false if both F and G are true, and true otherwise.

a. Write the corresponding truth-table.

b. Show that $\neg p$ can be defined by means of the Sheffer stroke.

c. Define $\vee$ in terms of $|$ (hint: compare the truth-tables).

d. Conclude that the Sheffer stroke is functionally complete.