

We did a lot yesterday...☺

- We related proof and truth

Proof	Truth
$\vdash F$	$\models F$
Trees	Assignments
Tree on $\neg F$ closed	F is valid
Tree on F open	F is satisfiable

...and even more... $\smile$

We discovered and related some central concepts of logic:

- decidability: PL is decidable
- completeness: PL has a sound and complete proof procedure
- compactness: when F follows logically from Γ , it follows from a finite subset of Γ .

We saw a crucial lemma: $\Gamma \models F$ iff $\Gamma \cup \{\neg F\}$ is not satisfiable.

Glimpses beyond

- There exist other methods of proof for propositional logic:
e.g., axiomatic systems (Frege), natural deduction (Gentzen, Prawitz), sequent calculus (Gentzen), resolution (Robinson).
- We won't have time to study them, but the important point is that they are not all analytic. However, one can establish close correspondences between these systems and the tree method.

Exercises

1. Use the tree method to determine whether the following formulae are tautologies:

$$((p \rightarrow q) \rightarrow p) \rightarrow p$$

$$((p \rightarrow \neg p) \rightarrow \neg p$$

$$((p \vee q) \rightarrow r) \rightarrow (p \rightarrow r) \wedge (q \rightarrow r)$$

$$((p \vee q) \rightarrow r) \rightarrow (p \rightarrow r)$$

2. (from van Dalen) Define the relation $F \prec G$ (F asymmetrically entails G) iff $\models F \rightarrow G$ but $\not\models G \rightarrow F$.
 - a. Give an example of two formulae F and G such that $F \prec G$
 - b. Construct an infinite sequence of formulae such that each formula in the sequence asymmetrically entails the next one.
 - c. Show that given two formulae F and G such that $F \prec G$, one can find a formula H such that $F \prec H \prec G$.
3. Show that the theory consisting of the single formula $p \vee q$, over the alphabet consisting only of p and q is incomplete (ie find a formula F of this language such that neither F nor $\neg F$ follows from that theory).

Goal of today

- Syntax and Semantics of FOL

4. First-order Predicate Logic

Expressive Limitation

- Propositional logic is very limited in expressive power

Everyone likes Mary $\rightsquigarrow p$

Mary is a painter $\rightsquigarrow q$

There is a painter whom everyone likes $\rightsquigarrow r$

- Yet, formally, $p, q \not\models r$.

- **Vocabulary**

Mary $\rightsquigarrow m$

x is painter $\rightsquigarrow P(x)$

x likes y $\rightsquigarrow L(x, y)$

- **Full sentences**

Everyone likes Mary $\rightsquigarrow \forall x L(x, m)$

Mary is a painter $\rightsquigarrow P(m)$

There is a painter whom everyone likes $\rightsquigarrow \exists y (P(y) \wedge \forall x L(x, y))$

Syntax of first-order logic

Vocabulary

- Individual variables: $x, y, z, x', y', z', \dots$
- Individual constants: $a, b, c, a', b', c', \dots$
- Predicate symbols: $P^{(1)}, Q^{(1)}, R^{(1)}, \dots, P^{(2)}, Q^{(2)}, R^{(2)}, \dots$
- connectives : $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
- Quantifier symbols: $\forall, \exists$

Formulae

1. Every variable or constant is a term.
2. Atomic formula: $R(t_1, \dots, t_n)$
3. Complex formulae: if ϕ and ψ are formula, so are $\neg\phi$, $\phi \wedge \psi$, $\phi \rightarrow \psi$, $\phi \vee \psi$, $\psi \leftrightarrow \psi$
4. If x is a variable and ϕ is a formula, $\forall x\phi$ and $\exists x\phi$ are formulae.
5. Nothing else is a formula.

Bound and Free variables

Compare:

(1) $\forall x R(x, y)$

(2) $\forall x \forall y R(x, y)$

Bound and Free variables

$$(1) \forall x R(x, y)$$

$$(2) \forall x \forall y R(x, y)$$

y is free in (1), in the scope of $\forall$ in (2).

Def (a bit informal). The free variables of a formula are the variables of the formula that are not bound by a quantifier.

Def. A sentence, or closed formula, is a formula without free variables.

(2) is a sentence ; (1) is an open formula.

First-order language

- A *language* is a set of predicate and constant symbols.
- ex : $L_1 = \{\leq; 0\}$

In L_1 , one can write formulae like:

$$\forall x(0 \leq x), \exists x \forall y(x \leq y)$$

- ex: $L_2 = \{P^{(1)}, L^{(2)}; m\}$: the language of our first example.

Aristotelian sentences

- Unrestricted quantification: $\forall xM(x)$, $\exists xM(x)$
- Restricted quantification

Every man is happy : $\forall x(M(x) \rightarrow H(x))$

Some man is happy : $\exists x(M(x) \wedge H(x))$

No man is happy : $\forall x(M(x) \rightarrow \neg H(x))$

Not every man is happy : $\neg \forall x(M(x) \rightarrow H(x))$

Exercise

Translate the following sentences:

1. No student knows all the professors
2. All the rich men like fishing.
3. John has a dog that he likes.

Interpretation

Given a language L , one interprets the formulas relative to an L -structure. An L -structure $M = (U, \mathcal{I}^M)$ consists of:

- a non-empty set of individuals U (universe of discourse)
- an interpretation function $\mathcal{I}^M$ mapping the constants and predicates of L to parts of U :
 - if c is a constant symbol, $\mathcal{I}^M(c) = c^M$ is an element in U .
 - if R is a predicate symbol of arity n , $\mathcal{I}^M(R) = R^M$ is a subset of U^n .

Example

- Consider the language $L = \{P, Q\}$, where P and Q are unary predicates.
- Take $M = (\mathbb{N}, \mathcal{I}^M)$, where $\mathcal{I}^M(P) = \{0, 2, 4, 6, \dots\}$ and $\mathcal{I}^M(Q) = \{1, 3, 5, 7, \dots\}$.
- M can be written $M = (\mathbb{N}, P^M, Q^M)$

Symbol	Interpretation
P	$P^M =$ the set of even numbers
Q	$Q^M =$ the set of odd numbers

Example-continued

Consider the formula $\phi = \forall x(P(x) \vee Q(x))$

Relative to the previous structure M , it is true, and it means:
“every integer is even or odd”. We will write:

$$M \models \forall x(P(x) \vee Q(x))$$

‘ M is a **model** of ϕ ’

‘ M satisfies ϕ ’

‘ ϕ is true in M ’.

- Models play the same role as truth-value assignments in propositional logic

Example - continued

However, let $M' = (\mathbb{N}, \mathcal{I}^{M'})$ where $\mathcal{I}^{M'}(P) = \{2, 4, 6, 8, \dots\}$, and $\mathcal{I}^{M'}(Q) = \{1, 3, 5, 7, \dots\}$.

Note that 0 is neither in $P^{M'}$ nor in $Q^{M'}$.

Hence : $M' \not\models \forall x(P(x) \vee Q(x))$

- The same sentence can be true in one structure, and false in another.
- Some sentences are true in every structure:

ex : $\psi = \forall x(P(x) \vee \neg P(x))$

Check that $M \models \psi$, $M' \models \psi$

Satisfaction and validity

- An L -sentence ϕ is satisfiable if there exists an L -structure M such that $M \models \phi$.
- An L -sentence ϕ is valid if it is satisfied in every L -structure M .

Notation: $\models \phi$.

However: we have not formally defined the notion of satisfaction yet...

NB. By satisfaction, we mean the same as truth

Two ways of interpreting variables

- A) Substitutional quantification
- B) Variable assignments

A) Substitutional quantification

Given a structure M , we add to the language a new name $\bar{c}$ for every element c of M . Call L' this enriched language. We allow M to interpret those new constants, so that $\bar{c}^M = c$.

$$M \models R(c_1, \dots, c_n) \text{ iff } (c_1^M, \dots, c_n^M) \in R^M$$

$$M \models \neg\phi \text{ iff } M \not\models \phi$$

$$M \models (\phi \wedge \psi) \text{ iff } M \models \phi \text{ and } M \models \psi$$

$$M \models \exists x\phi(x) \text{ iff } M \models \phi(\bar{c}) \text{ for some } \bar{c} \text{ of } L'.$$

$$M \models \forall x\phi(x) \text{ iff } M \models \phi(\bar{c}) \text{ for every } \bar{c} \text{ of } L'.$$

B) Variable Assignments

- The difficulty is to interpret variables. So far, our interpretations do not take care of variables, but only of the fixed vocabulary, namely constants and predicates.
- To interpret the variables, we will use an additional device, namely **assignments**. Given a structure $M = (U, \mathcal{I}^M)$, a variable assignment g assigns to each variable an element of the domain U .

Question: why not let the interpretation function do this?

Because, among other things, variables act like pronouns, which can change their value depending on the context.

Variables as pronouns

- The bound variable $\leftrightarrow$ anaphoric pronoun

(1) There is a painter whom everyone likes

$$\exists x(P(x) \wedge \forall yL(y, x))$$

There is someone such that **he** is a painter and everyone likes **him**

- Free variable $\leftrightarrow$ deictic pronoun

(2) **He** likes Mary

$$L(x, m)$$

Here the value of “he” depends on the context.

In context g , “he” denotes John, in context h , “he” denotes Luke.

Assignments

- Rule of thumb: think of variable assignments as contexts fixing the value of deictic pronouns!
- More than one pronoun needed:

John was observing Dave. He_[x:John] could see that he_[y:Dave] was not a good swimmer.

	x	y	z	...
g	John	Dave	Bill	...
h	Luke	Greg	John	...
j	Greg	Dave	Bill	...

- Given a context, we can interpret a sentence containing deictic pronouns: in context g , if L means “like”, $L(x, y)$ is true iff $g(x)$ likes $g(y)$.

- Given a structure $M = (U, \mathcal{I})$, and an assignment g , we note $t^{M,g} = \mathcal{I}(t)$ if t is a constant, and $t^{M,g} = g(t)$ if t is a variable.
- Given an assignment function g , we note $g[x : d]$ the assignment which is like g , except that it assigns to x the value d .
ex : on the previous slide, $g[x : Greg]$ corresponds to j .

Definition of satisfaction

$M, g \models R(t_1, \dots, t_n)$ iff $(t_1^{M,g}, \dots, t_n^{M,g})$ belongs to R^M .

$M, g \models \neg\phi$ iff $M, g \not\models \phi$

$M, g \models (\phi \wedge \psi)$ iff $M, g \models \phi$ and $M, g \models \psi$

$M, g \models \exists x\phi(x)$ iff there exists a d in U such that $M, g[x : d] \models \phi(x)$

$M, g \models \forall x\phi(x)$ iff for every d in U , $M, g[x : d] \models \phi(x)$

Satisfiability again

- A formula ϕ is satisfiable if there is an interpretation M and an assignment g such that $M, g \models \phi$.
- A formula ϕ is valid if for every interpretation M and assignment g , $M, g \models \phi$.
- If ϕ is a sentence, whenever $M, g \models \phi$ for some g , $M, g \models \phi$ for all g , so we can dispense with the reference to assignments and keep writing: $M \models \phi$.

Example

Show that $\forall x(P(x) \vee \neg P(x))$ is valid. Consider a structure M and an assignment g .

$$M, g \models \forall x(P(x) \vee \neg P(x))$$

iff for every d in U , $M, g[x : d] \models P(x) \vee \neg P(x)$

iff for every d in U , $M, g[x : d] \models P(x)$ or $M, g[x : d] \models \neg P(x)$.

iff for every d in U , d is in P^M or d is in P^M .

Logical consequence

Def. a sentence ψ is a logical consequence of ϕ iff every model of ϕ is also a model of ψ .

Notation : $\phi \models \psi$

Example : scope ambiguities

“Everyone loves someone” is ambiguous. Compare:

$\psi = \forall x \exists y L(x, y)$ (everyone loves someone or other)

$\phi = \exists y \forall x L(x, y)$ (someone is loved by everyone)

(a) Show that $\phi \models \psi$

(b) Show that $\psi \not\models \phi$

- (b) means : find M which satisfies ψ and not ϕ .

Look at the structure containing three individuals a, b, c , such that a loves b , b loves c , and c loves a . Clearly: everyone loves someone, but no one is loved by everyone.

- For (a) : suppose $M \models \exists y \forall x L(x, y)$. There is a constant, suppose it is $\bar{c}$, such that $M \models \forall x L(x, \bar{c})$. And for every constant d , $M \models L(\bar{c}, \bar{d})$. So for every $\bar{d}$, there is a constant, namely $\bar{c}$, such that $M \models L(\bar{c}, \bar{d})$. That is, $M \models \forall x \exists y L(x, y)$.

Some important validities

- Interdefinability of $\forall$ and $\exists$:

$$\models \exists x\phi \leftrightarrow \neg\forall x\neg\phi$$

$$\models \forall x\phi \leftrightarrow \neg\exists x\neg\phi$$

- Interaction with conjunction and disjunction

$$\models \forall x(P(x) \wedge Q(x)) \leftrightarrow (\forall xP(x) \wedge \forall xQ(x)) \text{ (everything is red and big)}$$

$$\models \exists x(P(x) \vee Q(x)) \leftrightarrow (\exists xP(x) \vee \exists xQ(x)) \text{ (something is red or big)}$$

exercise: show that $\not\models \forall x(P(x) \vee Q(x)) \rightarrow (\forall xP(x) \vee \forall xQ(x))$

Show that : $\models (\forall xP(x) \vee \forall xQ(x)) \rightarrow \forall x(P(x) \vee Q(x))$

Exercises

1. Consider the following FO-theory, over a language containing only the binary symbol R :

$$\forall \neg R(x, x)$$

$$\forall x \exists y R(x, y)$$

$$\forall x \forall y (R(x, y) \rightarrow \neg R(y, x))$$

a) Show that this theory is satisfiable in a model with 3 elements, but not in a model with only one, or two elements.

b) Show that the first axiom is entailed by the third.

c) We extend the theory with the axiom:

$$\forall x \forall y \forall z (R(x, y) \wedge R(y, z) \rightarrow R(x, z)).$$

Show that the new theory is satisfiable only in an infinite model.

The “drinker puzzle”

2. a) How would you translate the formula ($P(x) := x$ drinks)

(i) $\exists x(P(x) \rightarrow \forall yP(y))$?

b) Show that this formula is logically valid (reason by cases: either $\forall yP(y)$ holds, or not). Show that (ii) $(\exists xP(x) \rightarrow \forall yP(y))$ is not valid, and conclude the two formulae don't have the same meaning.

c) Show (by syntactic manipulations) that formula (i) is logically equivalent to (iii) $(\forall xP(x) \rightarrow \forall xP(x))$

d) Yet would you say that the formulae (i) and (iii) have the same intuitive meaning? Discuss the significance of this puzzle w.r.t the link between natural language and FOL.

The validity problem

Is there a sound and complete method for validity/satisfiability/logical consequence in predicate logic, as there is for propositional logic ?

Answer: YES, but there is no decision method.

Let's see with analytic trees...