Introduction to the Logic of Conditionals

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Lecture 2. Conditionals as restrictors

Reminder on Stalnaker's semantics

Let us review Stalnaker's semantics for "if ϕ then ψ", the core of all other non-monotonic conditional semantics:

- Either ϕ holds at the actual world: $w \models \phi$, and in that case, $w \models \phi > \psi$ iff $w \models \phi \rightarrow \psi$ (no adjustment needed)
- Or $\neg\phi$ holds at the actual world: $w \models \neg\phi$, and so $w \models \phi > \psi$ iff $f(\phi, w) \models \psi$ (adjust w minimally to make it consistent with ϕ)

Are conditionals binary connectives?

- In all accounts we considered so far, we have assumed that the conditional is a binary connective. Yet compare:

  (1) Always, if it rains, it gets cold.

  (2) Sometimes, if it rains, it gets cold.

- Can the “if"-clause be given a uniform semantic role?

Conditionals and coordination (Bhatt and Pancheva 2006)

- If-clauses can appear sentence-initially and sentence-finally. Not so with *and* and *or*:

(3) a. Joe left if Mary left.
b. If Mary left Joe left.

(4) a. Joe left and Mary left.
b. *and Mary left Joe left.

- Even if, only if:

(5) a. Lee will give you five dollars even if you bother him.
b. *Lee will give you five dollars even and you bother him.

Lycan 2001

- Conjunction reduction

(6) a. I washed the curtains and turned on the radio
b. *I washed the curtains if turned on the radio.

- Gapping

(7) a. I washed the curtains and Debra the bathroom.
b. *I washed the curtains if Debra the bathroom

- VP-ellipsis:

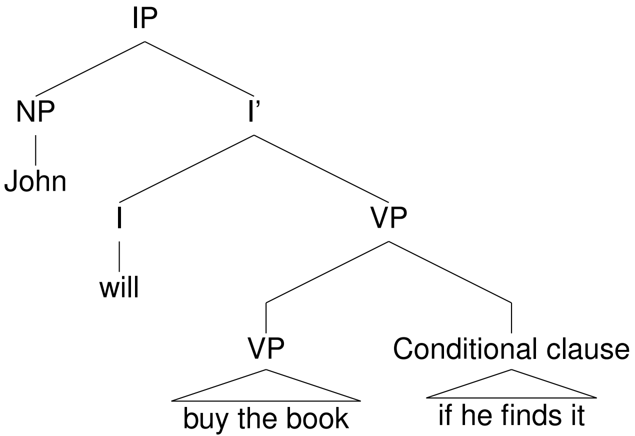
(8) I will leave if you do and John will ~~leave if you do~~, too

(9) I will leave if you do and John will *do so* too

"The data involving modification by only and even, and VP ellipsis phenomena provide strong evidence against the view that the antecedent and consequent of conditionals are coordinated. These data support the view that if-clauses are adverbials, like temporal phrases and clauses. Furthermore, pronominalization by then suggests that if-clauses are advervials, since their anaphoric reflex - then - is an adverb". Bhatt and Pancheva 2006

Syntactic representation

One possibility: from Haegeman 2003



Warning

The previous tree accounts only for the position of sentence-final if. The real story is more complicated (and left for syntacticians!):

> *Iatridou (1991) proposes that sentence-final if-clauses involve VP-adjunction, while sentence-initial if-clauses involve IP-adjunction. (Bhatt and Pancheva 2006)*

If-clauses as restrictors

A first try

(10) a. Always, if it rains, it gets cold.
b. $\forall t(R(t) \rightarrow C(t))$

(11) a. Sometimes, if it rains, it gets cold.
b. $\exists t(R(t) \rightarrow C(t))$.
c. $\exists t(R(t) \wedge C(t))$

Obviously, the intended reading is (11)-c, and (11)-b is simply inadequate. The intended reading is:

(12) Some cases in which it rains are cases in which it gets cold.

- Problem: how can “if" be given a uniform semantic role? What about other adverbs: *often*, *most of the time*,...?

Lewis 1975

- *If* as an adverbial restrictor

  the if of our restrictive if-clauses should not be regarded as a sentential connective. It has no meaning apart from the adverb it restricts (Lewis 1975: 14).

Kratzer 1991

"*The history of the conditional is the story of a syntactic mistake.*

- *There is no two-place if...then connective in the logical forms of natural languages.*
- *If-clauses are devices for restricting the domains of various operators.*
- *Whenever there is no explicit operator, we have to posit one."*
  (Kratzer 1991)

Monadic predicate logic with most

1. Atomic formulae: $Px, P'x, Qx, Q'x, ...$
2. Boolean formulae: $\phi := Px|\neg\phi|\phi \wedge \phi|\phi \vee \phi|\phi \rightarrow \phi$
3. Sentences: if ϕ, ψ are Boolean formulae: $\exists x\phi$, $\forall x\phi$, $Most\ x\phi$, as well as $[\exists x : \phi][\psi]$, $[\forall x : \phi][\psi]$, $[Most\ x : \phi][\psi]$.

Terminology: $[Qx : \phi(x)][\psi(x)]$:

- $\phi(x)$= quantifier restrictor
- $\psi(x)$ = nuclear scope

Semantics

Model: $M = \langle U, I \rangle$

1. $I(Px) \subseteq U$
2. $I(\phi \wedge \psi) = I(\phi) \cap I(\psi)$
   $I(\phi \vee \psi) = I(\phi) \cup I(\psi)$
   $I(\neg\phi) = \overline{I(\phi)}$
   $I(\phi \rightarrow \psi) = \overline{I(\phi)} \cup I(\psi)$
3. $M \models \forall x\phi$ iff $I(\phi) = U$
   $M \models \exists x\phi$ iff $I(\phi) \neq \emptyset$
   $M \models Most\ x\phi$ iff $|I(\phi)| > |I(\neg\phi)|$
4. $M \models [\forall x : \phi][\psi]$ iff $I(\phi) \subseteq I(\psi)$
   $M \models [\exists x : \phi][\psi]$ iff $I(\phi) \cap I(\psi) \neq \emptyset$
   $M \models [Most\ x : \phi][\psi]$ iff $|I(\phi) \cap I(\psi)| > |I(\phi) \cap \overline{I(\psi)}|$

The case of most

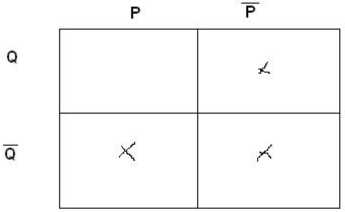
- $\models [\forall x : \phi][\psi] \leftrightarrow \forall x(\phi \rightarrow \psi)$
- $\models [\exists x : \phi][\psi] \leftrightarrow \exists x(\phi \wedge \psi)$

However:

- $\models Most\ x(\phi \wedge \psi) \rightarrow [Most\ x : \phi][\psi] \rightarrow Most\ x(\phi \rightarrow \psi)$
- But no converse implications.

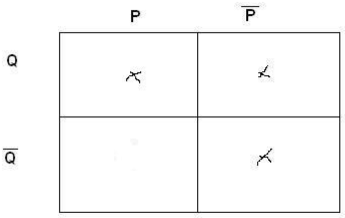
Most *x* are not P or Q $\not\Rightarrow$ Most Ps are Qs

- *Most* $x(Px \rightarrow Qx)$ is consistent with “no P is Q" (hence with $\neg[Most\ x : Px][Qx]$)



Most Ps are Qs $\not\rightarrow$ Most *x* are P and Q

- $[Most\ x : Px][Qx]$ is consistent with $\neg Most\ x(Px \wedge Qx)$.



Restricted quantification

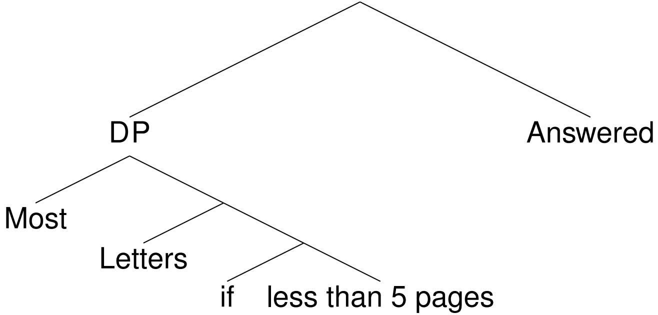
- Conclusion: the restrictor of most cannot be expressed by a material conditional (too weak) or a conjunction (too strong)
- Restricted quantification is needed to express if-clauses:

(13) a. Most of the time, if it rains, it gets cold.
b. $[Most\ t : Rt][Ct]$

(14) a. Most letters are answered if they are shorter than 5 pages.
b. Most letters that are shorter than 5 pages are answered.
c. $[Most\ x : Lx \wedge Sx][Ax]$.

Picture

- If-clauses usually attach a first quantifier restrictor, giving the effect of restrictive relative clauses.



Kratzer on conditionals

Conditional modality

How to analyze the interaction of a conditional with a modal?

(15) If a murder occurs, the jurors must convene. (in view of what the law provides)

Two prima facie candidates:

(16) a. $M \rightarrow \Box J$
b. $\Box(M \rightarrow J)$

Narrow scope analysis

Suppose that:

(17) a. No murder must occur. (in view of what the law provides)
b. $\Box\neg M$

Then, monotonicity problem:

(18) $\Box(\neg M \vee P)$ for every P.

In particular:

(19) $\Box(M \rightarrow J)$

Wanted: avoid automatic inference from “must $\neg p$" to “must if p, q" (a form of the paradox of material implication)

Wide scope analysis

(20) a. $M \rightarrow \Box J$
b. “no murder occurs, or the jurors must convene"

- Suppose a murder occurs. Then it should be an unconditional fact about the law that: $\Box J$, ie “the jurors must convene": too strong.

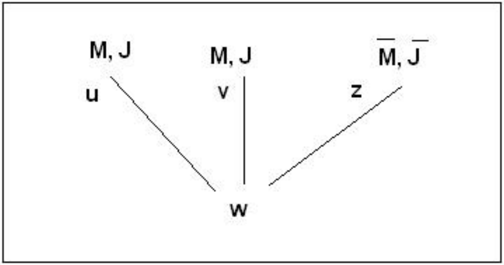
Illustration

Suppose there are two laws:

(21) a. If no murder occurs, the jurors are not allowed to convene.
b. $\neg M \rightarrow \Box \neg J$

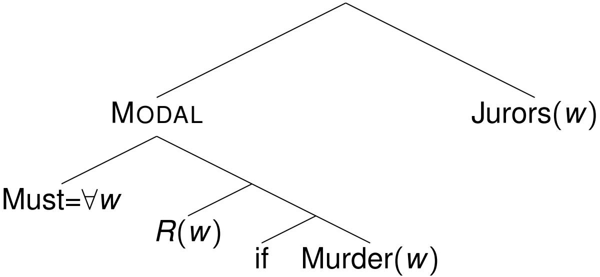
(22) a. If a murder occurs, the jurors must convene.
b. $M \rightarrow \Box J$

- Bad consequence: $\Box J \vee \Box \neg J$



- $M, w \not\models \Box J \vee \Box\neg J$, but intuitively: in all the worlds in which a murder occurs, the jurors convene, and in all the worlds in which no murder occurs, the jurors do not convene.

A syntactic improvement
(schema from von Fintel and Heim)



Problem: no semantic improvement on the strict conditional analysis when the modal is universal ("must") (but a semantic improvement with "might", "most of the time", ...)

Kratzer's solution: informally

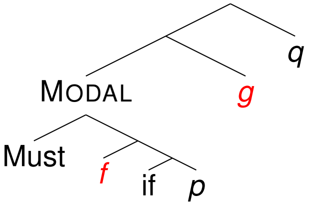
- $\Box p$: will be true not simply if p if true at all accessible worlds, but if p is true at all closest accessible worlds, or all ideal accessible worlds.
- Modality are doubly relative: to an accessibility relation, to an ordering.

Kratzer's solution: Doubly relative modalities

- Conversational background: *"a conversational background is the sort of entity denoted by phrases like what the law provides, what we know, etc. ... What the law provides is different from one possible world to another. And what the law provides in a particular world is a set of propositions...."*
- On Kratzer's analysis: 2 kinds of conversational backgrounds, modal base, and ordering source.

Schema

(from von Fintel and Iatridou 2004)



Correspondence with Stalnaker-Lewis:

- $f \equiv R(w)$
- $g \equiv \leq_w$

Modal base

- *The denotation of what we know is the function which assigns to every possible world the set of propositions we know in that world*
- Modal base: function f from W to $\wp(\wp(W))$ such that $f(w) = \{A, B, ...\}$
- By definition: $wR_f w'$ iff $w' \in \cap f(w)$

Ordering source

“there is a second conversational background involved... We may want to call it a stereotypical conversational background (“in view of the normal course of events"). For each world, the second conversational background induces an ordering on the set of worlds accessible from that world".

- Definition of an order: $\forall w, w' \in W, \forall A \subseteq \mathcal{P}(W) : w <_A w'$ iff $\{P; P \in A \wedge w' \in P\} \subset \{P; P \in A \wedge w \in P\}$
- A is a set of propositions: $w <_A w'$ iff w satisfies all propositions from A that w' satisfies and more besides.
- For each world, $g(w)$ picks such a set of propositions as an ordering source
- for $X \subseteq W$: $max_A(X)$=$\{w \in X | \neg\exists w' \in X : w' <_A w\}$

Kratzer's semantics for modals

- Model $M = \langle W, f, g \rangle$
- $w \models_{f,g} \Box\phi$ iff for all z such that $z \in max_{g(w)}(\cap f(w))$, $z \models_{f,g} \phi$
- ϕ is necessary iff it is true in all accessible worlds that come closest to the ideal.
- Remark: assumption that $<_{g(w)}$ always has some minimal elements (Limit Assumption)

Conditional modalities

- “if ϕ, then must ψ":=$(\Box : \phi)(\psi)$
- $w \models_{f,g} (\Box : \phi)(\psi)$ iff $w \models_{f',g} \Box\psi$, where $f'(w) = f(w) \cup \{ [\![\phi]\!]^{f,g} \}$.

“the analysis implies that there is a very close relationship between if-clauses and operators like must. They are interpreted together. For each world, the if-clause is added to the set of propositions the modal base assigns to that world. This means that for each world, the if-clause has the function of restricting the set of worlds which are accessible from that world".

Deontic conditional

(23) No murder must occur. If a murder occurs, the jurors must convene.

- Let: $g(w) = \{\neg M, M \rightarrow J\}$, and $f(w) = \emptyset$
- Let: $u \models \neg M$; $v \models M, J$; $z \models M, \neg J$. Then:

$$u <_{g(w)} v <_{g(w)} z$$

- $u \models_{f,g} \neg M$ and $\{u\} = max_{g(w)}(\cap f(w))$, so $w \models_{f,g} \Box\neg M$.
- $w \models_{f,g} (\Box : M)J$ iff $w \models_{f',g} \Box J$, where $f'(w) = f(w) \cup \{ [\![\mathrm{M}]\!]^{f,g} \} = \{\{v, z\}\}$. Clearly $v \in \cap f'(w) = \{v, z\}$ satisfies J and belongs to $max_{g(w)}(\cap(f'(w))$.

Bare conditionals

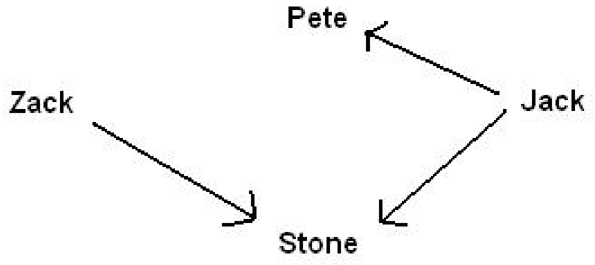
- According to Kratzer, bare conditionals are implicitly modalized
- Example discussed by Kratzer: epistemic modalities and conditionals.

Gibbard's riverboat example

Story (Gibbard 1981: 231): "Sly Pete and Mr. Stone are playing poker on a Mississipi riverboat. It is now up to Pete to call or fold. My henchman Zack sees Stone's hand, which is quite good, and signals its content to Pete. My henchman Jack sees both hands, and sees that Pete's hand is rather low, so that Stone's is the winning hand. At this point the room is cleared. A few minutes later, Zack slips me a note which says *if Pete called, he won*, and Jack slips me a note which says *if Pete called, he lost*.

- According to Kratzer: Zack's and Jack's utterances are both true.
- Why is it surprising? $\Diamond\phi \models_{Sta} (\phi > \psi) \rightarrow \neg(\phi > \neg\psi)$ (same with $\Box\!\!\rightarrow$ for Lewis).
- Each of them is true relative to a different modal base.

What Zack and Jack can see



A closer look

The difference is in Jack and Zack's modal bases

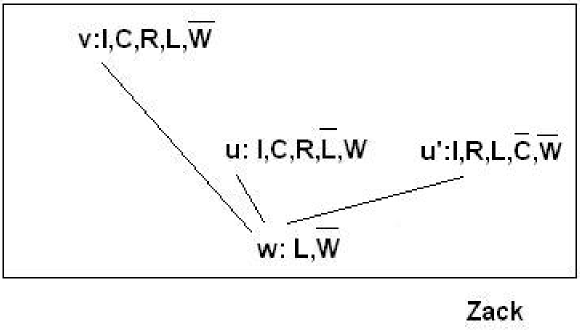
- $f_z(w) = \{$Pete is rational, Pete is informed about both hands$\} = \{R, I\}$
- $f_j(w) = \{$Pete is rational, Pete is informed about both hands, Pete's hand is lower than Stone's$\} = \{R, I, L\}$

A model for Sly Pete

Let: W=Pete will win.

- Common ordering source: $g_{j,z}(w) = \{RIC \rightarrow W, L \rightarrow \overline{W}\}$, ie nomic base includes: “a rational, informed player who calls wins, a lower hand does not win".
- For Zack: let $u \models I, C, R, \overline{L}, W$ and $v \models I, C, R, L, \overline{W}$, such that $\{u, v\} = \cap(f_z(w) \cup C)\}$. Then $u <_{g(w)} v$, and so $w \models_{f_z,g} (\Box : C)(W)$.
- For Jack: $\cap(f_j(w) \cup C)\} \subseteq L$ and does not contain any W-worlds. So $w \models_{f_j,g} (\Box : C)(\overline{W})$

Zack's state


v:I,C,R,L,$\overline{W}$
u: I,C,R,$\overline{L}$,W
u':I,R,L,$\overline{C}$,$\overline{W}$
w: L,$\overline{W}$
Zack

- Note 1: we stipulate that no possible world is L and W, ie no logically impossible possible world.
- Note 2: note that $\Box(C \rightarrow W)$ would not hold at w with a strict conditional analysis.

Comparisons

Are “if"-clauses always restrictors?

von Fintel/Iatridou 2002

(24) a. Most but not all of the students will succeed if they work hard
b. Most but not all of the students who work hard will succeed.

Scenario: 4 students: *a*, *b*, *c*, *d*. Suppose that if *a*, *b*, *c* are to work hard, they will all succeed. Suppose however that only *a* and *b* actually work hard, and both succeed.

- Then (24)-a is true, (24)-b false.

Observations

According to Iatridou and von Fintel, such examples “do not fall under an extension of the Lewis-Kratzer analysis".

- Fintel’s analysis: a strict conditional analysis with variable domain restrictions.
- $Q(x)[R(x)][\text{ if} w\ P(x,w), Q(x,w)]$

However: even when “if" does not restrict the quantifier “most", it can continue to restrict a hidden quantifier (à la Kratzer) if one assumes a strict conditional analysis.

Conclusion: maybe the problem is purely syntactic.

A variant with “exactly"

(25) Exactly half of the students got an A if they worked hard

Claim: 2 interpretations here as well

(26) a. Exactly half of the students who worked hard got an A
b. $[1/2 : S \wedge W][A]$

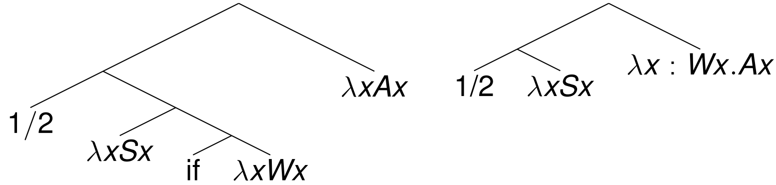
(27) a. Exactly half of the students got an A if those students worked hard.
b. $[1/2 : S][\text{if } W \text{ then } A]$.

Model

A model in which (26) is false and (27) true.

- $[\![S]\!] = \{a, b, c, d, e, f\}$
- $[\![W]\!] = \{a, b, c, d\}$
- $[\![A]\!] = \{a, b, c\}$.

Conditional as Presupposition



Let: $\lambda x : Wx.Ax$: function that takes value 1 if $Wx = Ax = 1$, value 0 if $Wx = 1$ and $Ax = 0$, undefined if $Wx = 0$.

“If" vs. “Necessarily if"

- Schlenker observes a contrast in monotonic behavior between:

(28) If the US got rid of all its weapons, there would be war. However, if the US and all other nuclear powers got rid of all their weapons, there should be peace.

(29) Necessarily, if the US got rid of all its weapons, there would be war. # However, if the US and all other nuclear powers got rid of all their weapons, there should be peace.

Modal strength

- Not necessarily an objection to Kratzer's thesis of implicit modalization: Kratzer can maintain that the strength of implicit modals is potentially lower than that of explicit modals:

(30) Ceteris paribus, if the US got rid of all its weapons, there would be war. But if the US and all nuclear powers got rid of all their weapons, there would be piece.

(31) Whatever happens, if the US got rid of all its weapons, there would be war. But if the US and all nuclear powers got rid of all their weapons, there would be piece.

- Strict conditionals drive monotonicity back.

Weak necessity

- At least good a possibility: $P \leq_{w,f,g} Q$ iff for all $u \in (\cap f(w) \cap Q)$ there is $v \in \cap f(w)$ such that $v \leq_{g(w)} u$ and $v \in P$.
- Better possibility: $P <_{w,f,g} Q$ iff $P \leq_{w,f,g} Q$ and not $Q \leq_{w,f,g} P$
- Weak necessity: $P <_{w,f,g} \overline{P}$
- $w_{f,g} \models \boxdot\phi$ iff $[\![\phi]\!] <_{w,f,g} \overline{[\![\phi]\!]}$

- Assume that relative to $g(w)$: $u \sim u' < v < z$, and $[\![\phi]\!] = \{u, v\}$, $[\![\neg\phi]\!] = \{u', z\}$. Then: $[\![\phi]\!] < [\![\neg\phi]\!]$. If u, u', v, z all belong to $\cap f(w)$, then ϕ is a weak necessity, but not a necessity (ie $w \not\models_{f,g} \Box\phi$).
- Clearly: possible to have $w \models_{f,g} (\boxdot : \phi)(\psi)$ and $w \not\models_{f,g} (\Box : \phi)(\psi)$.
- Benefit: treatment of expressions like "most likely"

Material conditional

- Let f such that for every w, $\cap f(w) = \{w\}$. (f is totally realistic= leaves no room for uncertainty)
- $w \models_{f,g} (\Box : \phi)(\psi)$ iff $\cap(f(w) \cup [\![\phi]\!]) \subseteq [\![\psi]\!]$.
- $\cap(f(w) \cup [\![\phi]\!]) = \{w\}$ if $w \models \phi$ and $= \emptyset$ if $w \not\models \phi$.
- $w \models_{f,g} (\Box : \phi)(\psi)$ iff $w \models \phi \wedge \psi$ or $w \not\models \phi$.

Strict conditional

- Suppose $f(w) = g(w) = \emptyset$ (all worlds accessible, all worlds equally close)
- For all $X \subseteq W$, $max_\emptyset(X) = X$
- Hence: $max_\emptyset(\cap(\emptyset \cup \{ [\![\phi]\!] \})) = [\![\phi]\!]$.
- $w \models_{\emptyset,\emptyset} (\Box : \phi)(\psi)$ iff $[\![\phi]\!]^{f,g} \subseteq [\![\psi]\!]^{f,g}$.

Probability

Explicit probability and conditionals

(32) The chances are n/m that if p then q

(33) The probability is n/m that if p then q

(34) If p, then the probability that q is n/m

Grice's Paradox

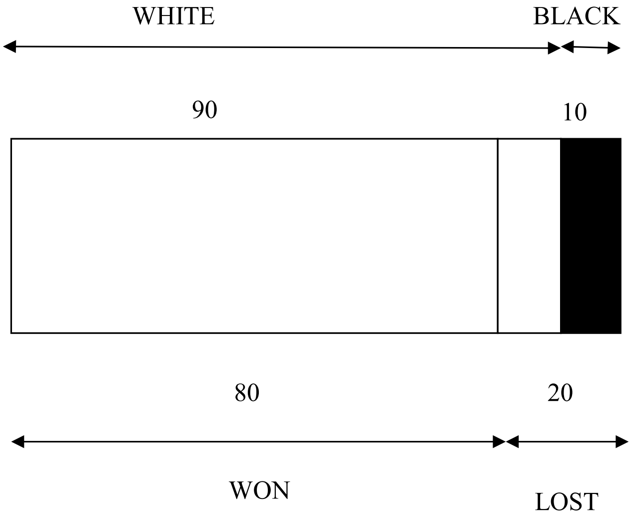
- Scenario: *Yog and Zog play chess. Up to now, there have been a hundred of games, no draws, and Yog had white 9 out 10 times. When Yog had white, he won 80 out of 90. When he had block, he lost 10 out of 10. We speak about the last game that took place last night and the outcome of which we do not know.*

(1) There is a probability of 8/9 that if Yog had white, he won

(2) There is a probability of 1/2 that if Yog lost, he had black

(3) There is a probability of 9/10 that either Yog didn't have white or he won

Yog and Zog's records



Grice's Paradox, cont.

- formalization:

(1) There is a probability of 8/9 that if Yog had white, he won

$$8/9(\mathit{WHITE} \Rightarrow \mathit{WON})$$

(2) There is a probability of 1/2 that if Yog lost, he had black $1/2(\mathit{LOST} \Rightarrow \mathit{BLACK})$

$$1/2(\neg \mathit{WON} \Rightarrow \neg \mathit{WHITE})$$

(3) There is a probability of 9/10 that either Yog didn't have white or he won

$$9/10(\neg \mathit{WHITE} \vee \mathit{WON})$$

Intuitive semantics

- Suppose a finite set of worlds
- $M, w \models [n/m](A)$ iff $\frac{|A|}{|W|} = n/m$
- $M, w \models [n/m : A][B]$ iff $\frac{|AB|}{|A|} = n/m$
- Agreement with Kratzer's thesis
- Grice's paradox explained away

Summary

- if-clauses as restrictors: allows a uniform semantic treatment of if-clauses with all quantifiers (modal and temporal operators/generalized quantifiers over individuals)
- the Lewis-Kratzer thesis, however, is in fact orthogonal to the issue of monotonicity
- A derivation of the notion of similarity ordering (a version of premise semantics, more in Lecture 5)
- A prima facie smooth treatment of interaction with probability operators: more to come... (Lectures 3 and 4)