Introduction to the Logic of Conditionals
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Lecture 3. Conditionals and Rational Belief Change

from semantics to epistemology

- during the last two lectures, we reviewed 5 semantical analyses of conditional sentences
- Lectures 3 and 4 will be much less linguistically oriented and will start from the epistemology of conditionals i.e. from rational belief attitudes towards conditionals
- more specifically, the main idea we will investigate: conditionals are closely linked to the dynamics of belief

conditionals and belief dynamics

- the main questions we will address are the following:

✓ how to elaborate and formalize the idea that conditionals are linked to belief dynamics ?

✓ is this idea useful for the semantics of conditionals ?

The Ramsey Test

the Ramsey Test

- the link between conditionals and the dynamics of belief is encapsulated in the famous **Ramsey Test (RT)**

  "If two people are arguing "If A will C ?" and are both in doubts as to A, they are adding A hypothetically to their stock of knowledge and arguing on that basis about C..." (F.P. Ramsey, 1929, "Law and Causality")

- Our reading of the (RT) : the belief attitude towards ($A \Rightarrow C$) is determined by the belief attitude towards C after a rational belief change on A

the Ramsey Test, illustration

- consider

(OI) If Oswald did not kill Kennedy, someone else did it

- how to evaluate my degree of confidence in (OI) ?

  • first I suppose that Oswald did not kill Kennedy (contrary to my actual beliefs)
  • then I revise my belief state in a minimal and rational way
  • lastly I evaluate "Someone else did kill Kennedy" in my new belief state

scope of the Ramsey Test

- as we will see, the Ramsey Test is much debated ; but even for those who accept it, it is generally assumed that it is not valid for all conditionals. People who are confident in the Ramsey Test claim typically that it is OK for *indicative conditionals*

(OI) If Oswald did not kill Kennedy, someone else did it

(OS) If Oswald had not killed Kennedy, someone else would have

- If I add to my present state of beliefs the proposition that Oswald did not kill Kennedy, I am pretty sure that someone else did it. This thought experiment converges to my intuitive evaluation of (OI), not of (OS).

some complications

- a sentence problematic for the Ramsey Test (from R.Thomason):

(1) If Reagan works for the KGB, I'll never find out

- but if I learned that Reagan works for the KGB, then I would have found out !
- several explanations/diagnoses:

  - conditionals and rational belief change go generally together, but there are complications with sentences dealing with belief attitudes

  - the Ramsey Test should be interpreted strictly: you have to suppose that the antecedent is true (not that it is true and that you *believe* that it is true !) (Stalnaker 1984)

two families of belief attitudes

- to elaborate formally on the Ramsey Test, one needs a model that captures both beliefs and rational belief change
- two kinds of (models of) beliefs:

(i) **full beliefs :** (a yes/no affair)
David believes that ϕ [acceptation]
David believes that $\neg\phi$ [reject]
David does not believe that ϕ [indeterminacy]

(ii) **partial beliefs :** (a matter of degree)
David believes that ϕ to degree *r*
David believes that it is probable that ϕ

two models of belief

- there are two main models of beliefand rational belief change :

(i) **belief revision theory** for full beliefs

(ii) **Bayesian probability** for partial beliefs

- *Ramsey Test for full beliefs* (RT_f) will be elaborated in the framework of belief revision (AGM etc.)
- *Ramsey Test for partial beliefs* (RT_p) will be elaborated in the framework of subjective probability. It will be equated (roughly) to a famous thesis, Adams Thesis (AT), equally named by Hajek & Hall (1994) the Conditional Construal of Conditional Probability (CCCP).

the belief revision framework

- Ramsey Test in a dynamic framework for full beliefs
- full beliefs are represented by **beliefs sets** K i.e. sets of formulae that satisfy certain rationality constraints
- let the language be $\mathcal{L}_3^{\Rightarrow}$ = the **full conditional language**
  $\phi := p \mid \neg\phi \mid \phi \vee \phi \mid \phi \Rightarrow \phi$
- a relation of logical consequence $\models$ is given on $\mathcal{L}_3^{\Rightarrow}$ that extends the classical one

the belief revision framework, cont.

- rationality constraints: belief sets (i) include all tautologies and (ii) are closed under $\models$
- the agent starts with an initial belief set K ; he or she receives a *message* represented by a formula ϕ and then revises his or her beliefs ; a new belief set $K * \phi$ is induced
- given a set of belief sets **K**, a **belief revision model** is a pair $(\mathbf{K}, *)$ where $* : \mathbf{K} \times \mathcal{L}_3^{\Rightarrow} \rightarrow \mathbf{K}$ is a revision operator (i.e. $K * \phi = *(K, \phi)$)

the Ramsey Test in belief revision

- in this framework, a weak reading of the Ramsey Test would be this one:

$$(\mathrm{WRT}_f)\ A \Rightarrow C \in K \text{ if } C \in K * A$$

- the received reading (e.g. Gärdenfors 1986) is a stronger one:

$$\boxed{(\mathrm{RT}_f)\ A \Rightarrow C \in K \text{ iff } C \in K * A}$$

The converse direction of (WRT_f) is called Conditional Driven Revision (CDR) by Bradley (2007) since it says that acceptation of a conditional commits to a revision policy

(CDR) and counterfactuals

- an example by Stalnaker (1984) shows that (CDR) is troublesome for *subjunctive conditionals*

(2) If Hitler had decided to invade England in 1940, Germany would have won the war ($A \Rightarrow C$)

(3) Hitler did decide to invade England in 1940 (A)

I give up $A \Rightarrow C$ rather than endorse C

rationality postulates

- (RT_f) won't give us anything interesting if we don't make assumptions on the revision operator
- the rationality constraints on $*$ have been intensively studied by belief revision theory
- examples:

  ✓ the postulate (K*2) $\phi \in K * \phi$ (Success)
  ✓ the postulate (K*5) $K * \phi = K_\perp$ iff $\models \neg\phi$ (Consistency)
  where $K_\perp$ is the absurd belief set that contains every sentences
- the paradigm view is the **AGM system** that includes 8 such postulates

the AGM postulates

(K*1) $K * \phi$ is a belief set
(K*2) $\phi \in K * \phi$
(K*3) $K * \phi \subseteq Cn\{K \cup \{\phi\}\} = K + \phi$
(K*4) If $\neg\phi \notin K$, $K + \phi \subseteq K * \phi$
(K*5) $K * \phi = K_\perp$ iff $\models \neg\phi$
(K*6) If $\models \phi \leftrightarrow \psi$, then $K * \phi = K * \psi$
(K*7) $K * (\phi \wedge \psi) \subseteq (K * \phi) + \psi$
(K*8) If $\neg\psi \notin K * \phi$, $(K * \phi) + \psi \subseteq K * (\phi \wedge \psi)$

a logic for conditionals based on the Ramsey Test

- (RT_f) does not give us truth-conditions for $\Rightarrow$
- but we can base a *logic* on it by considering which formulae are valid in every belief revision system
- the logic will of course depend on the rationality postulates

  ✓ $A \Rightarrow A$ will e.g. be valid in b.r.m satisfying (RT_f) and (K*1): since $A \in K * A$, by (RT_f) $A \Rightarrow A \in K$.
- question: which logic is induced by the AGM system ?

from the Ramsey Test to Lewis and Stalnaker

- two of the AGM postulates are problematic for conditional logics (essentially (K*4) - more on this tomorrow)
- if we put these two postulates aside, the logic obtained is equivalent to Lewis system **VC** - see Lecture 1
- if we add the postulate according to which

  (K*C) If K is maximal (i.e. $\forall A$, A or $\neg A \in K$) then $K * \phi$ is maximal as well

  one obtains a logic equivalent to Stalnaker system **C2**
- a first convergence of the Ramsey Test and Lewis-Stalnaker logics
  (for an exhaustive survey of this literature, see Nute & Cross (2001))

Conditional probability

subjective probabilities

- the main model of partial beliefs and change of partial beliefs is the Bayesian probabilistic model
- two central tenets:

(1) static: the belief state of a rational agent at some time is represented by a probability measure - his or her degrees of beliefs obey the laws of probability

(2) dynamic: when a rational agent learns an information, he changes his doxastic state by the so-called rule of conditionalization (more on this later)

probability functions

- let $\mathcal{L}$ be a language (a set of formulas) based (notably) on boolean connectives and $\models$ a consequence relation for that language (classical for boolean connectives)
- $P : \mathcal{L} \rightarrow \mathbb{R}$ is a **probability function** iff the following axioms are satisfied :

(P1) $0 \leq P(\phi) \leq 1$
(P2) If $\models \phi$, then $P(\phi) = 1$
(P3) If $\{\phi, \psi\} \models \bot$, then $P(\phi \vee \psi) = P(\phi) + P(\psi)$
(P4) If $\phi \equiv \psi$, then $P(\phi) = P(\psi)$

probability distributions

- a probability function is a syntactic version of the usual notion of probability distribution defined on algebras of subsets
- Let $(W, \mathbf{E})$ be a measurable space. $p : \mathbf{E} \rightarrow \mathbb{R}$ is a **probability distribution** on $(W, \mathbf{E})$ if $\forall\ E, E' \in \mathbf{E}$

(P1') $0 \leq p(E) \leq 1$
(P2') $p(W) = 1$
(P3') If $E \cap E' = \emptyset$, then $p(E \cup E') = p(E) + P(E')$

- for a finite state space $W = \{w_1, ..., w_n\}$, one can see a probability distribution on $(W, 2^W)$ as a function $p : W \rightarrow \mathbb{R}$ s.t.

$$\sum_{w_i \in W} p(w_i) = 1$$

linking the two notions

- semantics allows us to link these two notions. Given a model (W, I), consider $[\![\phi]\!] : \{w \in W : W, w \models \phi\}$. Let p be a probability distribution on W.
- then you may define $P(\phi) := p([\![\phi]\!])$ (the probability of ϕ is the probability of its truth)
- if I is a classical interpretation, then given an appropriate algebra $P(.)$ will obey (P1)-(P4)

some basic properties of probability functions

- Let $\phi, \psi \in \mathcal{L}$;

(1) $P(\neg\phi) = 1 - P(\phi)$

(2) If $\models \neg\phi$, then $P(\phi) = 0$

(3) $P(\phi) + P(\psi) = P(\phi \vee \psi) + P(\phi \wedge \psi)$

(4) $P(\phi) = P(\phi \wedge \psi) + P(\phi \wedge \neg\psi)$ (Addition Theorem)

- proof:
  (i) ϕ and $((\phi \wedge \psi) \vee (\phi \wedge \neg\psi))$ are equivalent
  (ii) $\phi \wedge \psi$ and $\phi \wedge \neg\psi$ are incompatible
  (iii) by (P3) and (P4), Addition Theorem

why should degrees of beliefs obey probabilities?

- the representation of a belief state by a belief set is coarse-grained but may seem much more intuitive than its representation by a probability function: **why should degrees of belief obey the laws of probability?**
- a first answer: think twice, it's self-evident !
- a more constructive answer: practical rationality requires beliefs as far as they guide action to obey probabilities = **Dutch Book Argument** Dutch Book

conditional probability

- the crucial notion in probabilistic belief dynamics is that of **conditional probability** $P(\psi|\phi)$
- $P(\psi|\phi)$ is defined (?) by the *Ratio Formula* (sometimes called the Quotient Rule):

$$P(\psi|\phi) = P(\phi \wedge \psi)/P(\phi) \text{ if } P(\phi) > 0$$

- **Fact**: for any probability function P, for any ϕ s.t. $P(\phi) > 0$, $P(.|\phi)$ is a probability function
- as a consequence, one can view
  • P as an initial or *a priori* doxastic state
  • $P(.|\phi)$ as an *a posteriori* doxastic state, that results from the learning ϕ - conditionalizing or learning by conditionalization

meaning of conditional probability

- the basic intuition behind the Ratio Formula : the probability of ψ given ϕ is the proportion of the ψ-worlds among the ϕ-worlds
- the meaning of the ratio formula can be grasped by considering the evolution of a world w after the information that ϕ is the case (i.e. that the actual world is among the $[\![\phi]\!]$-worlds):

(i) if $w \notin [\![\phi]\!]$ (i.e. $\delta_\phi(w) = 0$), $P(\{w\}|[\![\phi]\!]) = 0$

(ii) if $w \in [\![\phi]\!]$ (i.e. $\delta_\phi(w) = 1$), w's weight is normalized w.r.t. the total weight of $[\![\phi]\!]$-worlds :

$$P(\{w\}|[\![\phi]\!]) = P(\{w\})/\sum_{w' \in W}[\delta_\phi(w').P(\{w'\})]$$

an example

- initial probability P

w	ONE	TWO	THREE	FOUR	FIVE	SIX
P	1/6	1/6	1/6	1/6	1/6	1/6

- after conditionalization on $EVEN$, $P' = P(.|EVEN)$

w	~~ONE~~	TWO	~~THREE~~	FOUR	~~FIVE~~	SIX
P'	0	1/3	0	1/3	0	1/3

meaning of conditional probability, cont.

- **Fact**: if w and w' are compatible with ϕ, then

$$\frac{P(\{w\})}{P(\{w'\})} = \frac{P(\{w\}|\phi)}{P(\{w'\}|\phi)}$$

- this last property (which can be slightly generalized) gives a sense in which conditionalizing is a minimal way of changing (probabilistic) beliefs:

  "Conditionalizing P on A gives a minimal revision in this...sense: unlike all other revisions of P to make A certain, it does not distort the profile of probability ratios, equalities, and inequalities among sentences that imply A" (Lewis 1976)

conditionalization and invariance

- let's consider conditionalization as a probabilistic change rule of type $\mathbf{P} \times \mathcal{L} \rightarrow \mathbf{P}$
- an obvious property of conditionalization is **Certainty**: the new probability of the information is 1, $P_\phi(\phi) = 1$
- a less obvious property is **Invariance** of conditional probability: $\forall \psi \in \mathcal{L}$, $P(\psi|\phi) = P_\phi(\psi|\phi)$
- Jeffrey has noticed that Certainty + Invariance *characterize* conditionalization as a probabilistic change rule

conditionalizing as rational change of beliefs

- it is often claimed that conditionalizing is *the* rational way of changing one's partial beliefs
- the main kind of argument proposed to vindicate this claim is a *pragmatic justification* based on Dutch Books: it is named the *Dynamic Dutch Book* argument (Lewis - Teller)
- the DDB shows that if you don't update your beliefs by conditionalizing, then a clever bookie can devise a set of bets s.t. you will be willing to accept each of them whereas they collectively lead you to a sure loss

two theorems on conditional probabilities

- before coming back to conditionals, let me remind you two elementary theorems concerning conditional probabilities that will be useful in the sequel:

(i) **Bayes Theorem**:
$P(\psi|\phi) = [P(\phi|\psi).P(\psi)]/P(\phi)$

(ii) **Expansion By Case Theorem**:
$P(\phi) = P(\phi|\psi) \times P(\psi) + P(\phi|\neg\psi) \times P(\neg\psi)$
(follows from the Addition Theorem and the Ratio Formula)

Adams Thesis

Ramsey Test and Adams Thesis

- let's come back to the Ramsey Test and let P_ϕ denote a rational belief change based on the information that ϕ is the case
- a weak reading of (RT) for partial beliefs is this one: (WRT_p) If $P_A(C) = 1$, then $P(A \Rightarrow C) = 1$
- the received reading is a stronger one according to which

$$(RT_p)\ P(A \Rightarrow C) = P_A(C)$$

Ramsey Test and Adams Thesis, cont.

- the full quotation of (Ramsey, 1929) is nevertheless more specific:
  *"If two people are arguing "If p will q ?" and are both in doubts as to p, they are adding p hypothetically to their stock of knowledge and arguing on that basis about q ;... **they are fixing their degrees of belief in** q **given** p"*
- this specification of (RT_p) is known in the literature as **Adams Thesis** (E.W. Adams, *The logic of conditionals*, 1975) :
  "The fundamental assumption of this work is : the probability of an indicative conditional of the form "if A is the case then B is" is a conditional probability."

Adams Thesis

- Adams Thesis can be formulated in this way :

$$\text{(AT) } P(A \Rightarrow C) = P(C|A) \text{ if } P(A) > 0$$

- note that (AT) follows from (RT_p) + Conditionalization, but that it can be endorsed independently of any story about belief change
- (AT) is very compelling if we look at our intuitive judgements. Consider for instance the throw of a fair dice. What is the probability of

(4) if an even number comes up, the 6 comes up

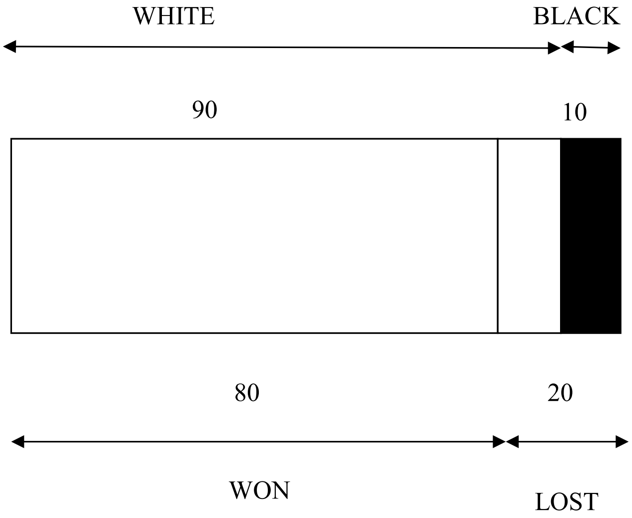
Grice's Paradox

- Scenario: *Yog and Zog play chess. Up to now, there have been a hundred of games, no draws, and Yog had white 9 out 10 times. When Yog had white, he won 80 out of 90. When he had block, he lost 10 out of 10. We speak about the last game that took place last night and the outcome of which we do not know.*

(1) There is a probability of 8/9 that if Yog had white, he won

(2) There is a probability of 1/2 that if Yog lost, he had black

Yog and Zog's records



material conditional and conditional probability

- it is easy to see how (AT) can contribute to the discussion on the semantics of the conditional. Consider the same example and the material conditional $(A \rightarrow C)$.

  What is the probability that either an even number doesn't come up or the 6 comes up ? 2/3

- the material conditional (1) doesn't conform to our intuitions on the probability of conditionals and (2) doesn't conform to Adams Thesis
- **Fact**: $P(\phi \rightarrow \psi) = P(\neg A) + P(A \wedge C) \ > \ P(\psi|\phi)$ except when $P(\phi \wedge \neg\psi) = 0$ or $P(\phi) = 1$ where $P(\phi \rightarrow \psi) = P(\psi|\phi)$

conditional probability and conjunction

- another consequence of (AT): $P(\phi \Rightarrow \psi) \geq P(\phi \wedge \psi)$ (strict inequality if $P(\phi) < 1$)
- if David believes strongly that $(\phi \wedge \psi)$, he believes at least as strongly that $(\phi \Rightarrow \psi)$
- to sum up:

$$\boxed{P(\phi \wedge \psi) \leq P(\phi \Rightarrow \psi) \leq P(\phi \rightarrow \psi)}$$

overview

$$\boxed{P(\phi \wedge \psi) \leq P(\phi \Rightarrow \psi) \leq P(\phi \rightarrow \psi)}$$

- compare with the Stalnaker-Lewis conditional: $(\phi \wedge \psi) \models (\phi > \psi) \models (\phi \rightarrow \psi)$ therefore for any P,

$$\boxed{P(\phi \wedge \psi) \leq P(\phi > \psi) \leq P(\phi \rightarrow \psi)}$$

(see as well
$\models \mathit{Most}\ x(\phi \wedge \psi) \rightarrow [\mathit{Most}\ x : \phi][\psi] \rightarrow \mathit{Most}\ x(\phi \rightarrow \psi))$

(AT) and Invariance

- remember: conditionalization is the belief change rule that obeys Certainty and Invariance of conditional probability
- putting this characterization together with (AT), this means that *the probability of a conditional $A \Rightarrow C$ doesn't evolve when you learn that the antecedent A is true* - the **invariance** of $\Rightarrow$
- not so for the material conditional: $P_A(A \rightarrow C) = P(C|A)) \leq P(A \rightarrow C)$ ▸ proof - you will necessarily lose confidence in the mat.cond when you learn that its antecedent is true !
- we will see later how F.Jackson exploits all this...to defend a truth-functional account of indicative conditionals

a counterexample to invariance

- invariance seems to be a compelling consequence of (AT). But see the following scenario (adapted from McGee 2000)
- you are quite confident in your banker who told you:

(5) Don't be anxious. The prices won't continue to go down ($\neg A$). And (even) if they continue, this won't have much impact on you ($A \Rightarrow C$).

- then you learn that the prices continue to go down. So it seems that your banker is not reliable after all. As a consequence, your degree of credence in $A \Rightarrow C$ (if the prices continue to go down, this won't have much impact on you) decreases.
- so $P_A(A \Rightarrow C) < P(A \Rightarrow C)$!

(AT) and null-probability antecedents

- the conditional probability $P(\psi|\phi)$ is not defined when $P(\phi) = 0$
- this seems to be troublesome for *counterfactual* conditionals: for most of these, the antecedent is supposed to be known to be false
- two main reactions:

(i) modify something in Adams Thesis so that the modified Thesis can work even for null-probability antecedents (see Stalnaker 1970 based on Popper functions)

(ii) restrict Adams Thesis to indicative conditionals *and* consider that indicative conditionals are “zero-intolerant” i.e. do not really make sense when the antecedent is supposed to be false

why believe Adams Thesis ?

- main arguments

1. adequacy with intuitive judgments about probability of conditionals (look at the example of the fair dice). (this is expected by the Lewis-Kratzer conditionals-as-restrictors view exposed yesterday, more on this tomorrow)
2. (AT) can be derived from (RT_p) + Conditionalization
3. Conditional Dutch Book more
4. from (AT), a logic of conditionals, **Adams logic**, can be built that represents quite faithfully the intuitively valid patterns of arguments involving conditionals

Adams Logic

Adams Logic: the language

- Let $\mathcal{L}_0$ be a propositional language (so-called **factual formulas**). The set of formulas $\mathcal{L}_1^{\Rightarrow}$ is given by
  - $F := p \mid \neg F \mid F \vee F$ (**factual formulas** = $\mathcal{L}_0$)
  - $\phi := F \mid F \Rightarrow F$ (**simple conditional formulas**)
- strong syntactic restriction: no embedding with conditionals, which appear only as main connectives.
- given a probability function P on $\mathcal{L}$ we know how to assign values to simple conditionals through (AT) and therefore how to extend P to $\mathcal{L}_1^{\Rightarrow}$.

Adams Logic: the language

- by contrast, (AT) does not tell us how to extend P defined initially on $\mathcal{L}$ to compounds of conditionals
  example : what is $P(B \wedge (A \Rightarrow C))$?
- if we were starting from a truth-conditional *semantics* for $\Rightarrow$ that assigns to each conditional $A \Rightarrow C$ a proposition $[\![A \Rightarrow C]\!]$ (e.g. Stalnaker-Lewis semantics), there would be no issue. In this case:
  $p([\![B \wedge (A \Rightarrow C)]\!]) = p([\![B]\!] \cap [\![A \Rightarrow C]\!])$
- caution: P extended to $\mathcal{L}_1^{\Rightarrow}$ is strictly speaking no longer a probability function
  For instance, it is not the case that
  $P(A \Rightarrow C) = P((A \Rightarrow C) \wedge B) + P((A \Rightarrow C) \wedge \neg B)$ since these two conjunctions are not in the language

probabilistic validity

- Adams develops a relation of logical consequence for $\mathcal{L}_1^{\Rightarrow}$ based on (AT)
- with (AT) alone - without truth-conditions for $\Rightarrow$ - one cannot develop the usual notion of Truth Preservation
- Adams idea: substituting **Probability Preservation** to Truth Preservation
- a second motivation (beside the lack of truth-conditions): outside mathematics, inferences are typically applied to less-than-certain premises

probabilistic validity

how to implement the Probability Preservation criterion ?

- option 1: **Certainty Preservation**: $\Gamma \models \phi$ iff $\forall P$, if $\forall \psi \in \Gamma$, $P(\psi) = 1$ then $P(\phi) = 1$ - this would fit the first motivation (we don't have truth-conditions for $\Rightarrow$) but not the second one (we typically reason from less-than-certain premises)
- option 2: **High Probability Preservation** criterion according to which $\Gamma \models \phi$ iff $\forall P$, ϕ is (roughly) at least as probable as the premises - this would fit the two motivations and this is the idea endorsed by Adams

probabilistic validity, cont.

here is the official notion of p-validity (for "probabilistic validity"):

- $\psi \models_p \phi$ iff for no prob.func. P, $P(\phi) < P(\psi)$ iff for all P

$$U(\phi) \leq U(\psi)$$

where $U(\phi) =_{df} 1 - P(\phi)$

- general case: $\Gamma \models_p \phi$ iff for all P

$$\boxed{U(\phi) \leq U(\psi_1) + ... + U(\psi_n)}$$

probabilistic validity, cont.

- note that it can be the case that $\Gamma \models_p \phi$ and $P(\phi)$ is very low even if for each individual premise $\psi_i \in \Gamma$, $P(\psi_i)$ is very high
- this is as it should be. See for instance the **Lottery Paradox**: a lottery with one winning ticket among 100 available. Let ψ_i $(1 \leq i \leq 100)$ mean "ticket $\sharp i$ is not the winning ticket" and let $\phi = \bigwedge_i \psi_i$
- $P(\psi_i) = 99/100$ but $P(\phi) = 0$

classical validity and *p*-validity

- **Proposition**: if $\Gamma \cup \{B\} \subseteq \mathcal{L}$ and $\Gamma \models_{PL} B$, then $\Gamma \models_p B$

  Proof:
  Assume $\{A_1, ..., A_n\} \models_{PL} B$
  (1) $\neg B \models_{PL} \neg A_1 \vee ... \vee \neg A_n$
  (2) $P(\neg B) \leq P(\neg A_1 \vee ... \vee \neg A_n)$ (since P is a probability function)
  (3) $P(\neg B) \leq P(\neg A_1) + ... + P(\neg A_n)$
  (4) $U(B) \leq U(A_1) + ... + U(A_n)$
  (5) $\{A_1, ..., A_n\} \models_p B$ ♠
- the converse is true as well; so for factual formulas, classical validity and *p*-validity coincide

$\neg A \therefore A \Rightarrow B$ is not *p*-valid

- failure of the paradox of the falsity of the antecedent

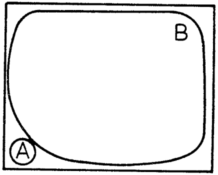


Fig. 2.

A = "It will not rain in Berkeley next year"
B = "It will rain in Berkeley next year"

$B \therefore A \Rightarrow B$ is not *p*-valid, cont.

- the same example can be used to illustrate the failure of the paradox of the truth of the consequent

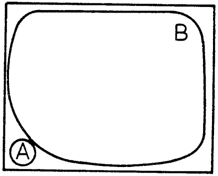


Fig. 2.

A = "It will not rain in Berkeley next year"
B = "It will rain in Berkeley next year"

$\rightarrow$ and $\Rightarrow$

- it is not the case that $A \rightarrow C \models_p A \Rightarrow C$
- but $P(A \rightarrow C) = 1$ implies $P(A \Rightarrow C) = 1$ (and conversely)
  If I am certain that the material conditional is true, I am certain that if A, C.
- clearly $P(A \rightarrow C) = 0$ implies $P(A \Rightarrow C) = 0$.
  $\rightarrow$ and $\Rightarrow$ collapse in the certain case.

$\{(A \Rightarrow B), (B \Rightarrow C)\} \therefore (A \Rightarrow C)$ is not p-valid

- failure of **Transitivity**:

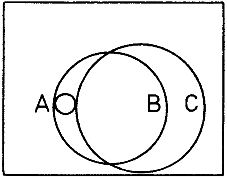


Fig. 5.

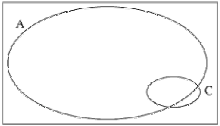
A = Smith will die before the election
B = Jones will win the election
C = Smith will retire after the election another example ?

$(B \rightarrow C) \therefore ((A \wedge B) \rightarrow C)$ is not *p*-valid

- but the following restriction is *p*-valid :
  $\{(A \Rightarrow B), ((A \wedge B) \Rightarrow C)\} \models_p (A \Rightarrow C)$
- the same diagram is a counterexample to **Antecedent Strenghtening**:
  ✓ $B \Rightarrow C$: If Jones wins the election, Smith will retire after the election
  × $(A \wedge B) \Rightarrow C$: If Smith dies before the election and Jones wins the election, Smith will retire after the election

$(A \Rightarrow \neg C) \therefore (C \Rightarrow \neg A))$ is not *p*-valid

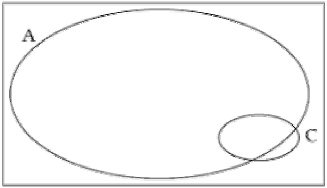
- failure of **Contraposition**: David is a movie critic and sees most of the movies. His friend Paul sees few movies, and most (but not all) of those he sees are seen by David as well. Let's consider a randomly chosen movie entitled *Life of a Logician*



It is likely that if David saw *Life of a Logician*, Paul didn't see it ($A \Rightarrow \neg C$)
It is unlikely that if Paul saw *Life of a Logician*, David didn't see it ($C \Rightarrow \neg A$)

$(A \vee C) \therefore (\neg A \Rightarrow C))$ is not p-valid

- failure of **Disjunctive Syllogism** (Bennett 2003):



A: There will be snow in Buffalo in 2009
C: A woman will be elected President of the USA in 2008

p-validity and plausibility

- in Transitivity, Monotony, DS and Contraposition, *conclusions are conditionals* ; and in all our counter-examples *the probability of the conclusions' antecedents are very low*
- that's not a coincidence:
(i) (ceteris paribus) the lower is the probability of the antecedent A, the more the probability of $A \Rightarrow C$ diverges from the one of $A \rightarrow C$
(ii) these inferences are *p*-valid with $\rightarrow$
- by contrast, when $P(A)$ is high, these inferences will look perfectly acceptable since $U(A \Rightarrow C) \cdot P(A) = U(A \rightarrow C)$. This is how Adams explains that for indicative conditionals they may appear as OK in general

a deductive system for Adams Logic

- here is a deductive system **ADS** for Adams Logic. A, B, C are factual formulas, ϕ, ψ are any formula

(R1) A premise ϕ can be stated on any line
(R2) If $A \vdash_{PL} B$, $\psi \mathrel{.\dot{.}.} A \Rightarrow B$ for any ψ
(R3) $(\top \Rightarrow A) \mathrel{.\dot{.}.} A$ and $A \mathrel{.\dot{.}.} (\top \Rightarrow A)$
(R4) If $A \equiv_{PL} B$, $A \Rightarrow C \mathrel{.\dot{.}.} B \Rightarrow C$
(R5) $\{A \Rightarrow C, B \Rightarrow C\} \mathrel{.\dot{.}.} (A \vee B) \Rightarrow C$
(R7) $\{A \Rightarrow B, (A \wedge B) \Rightarrow C\} \mathrel{.\dot{.}.} A \Rightarrow C$ (Restricted T)
(R8) $\{A \Rightarrow B, A \Rightarrow C\} \mathrel{.\dot{.}.} (A \wedge B) \Rightarrow C$ (Restricted M)

- **Completeness Theorem**: $\Gamma \models_p \phi$ iff $\Gamma \vdash_{ADS} \phi$

Adams conditional and Stalnaker Conditional

- let us pause and compare (i) the inference patterns accepted and rejected by *p*-entailment and (ii) those accepted and rejected by Stalnaker semantics. They are pretty much the same !
- one of the most striking facts of the early contemporary research on conditionals is that *APL and Stalnaker Conditional coincide on their common domain* (Stalnaker 1970 ; Gibbard, 1980) : if $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_1^{\Rightarrow}$ and Γ finite, then

$$\Gamma \models_p \phi \text{ iff } \Gamma \models_{Sta} \phi$$

idea of the proof

- the detailed proof of this impressive result is beyond the scope of this lecture
- if $\Gamma \models_p \phi$, then $\Gamma \models_{Sta} \phi$: can be proved by showing that Adams' deductive system is sound for Stalnaker semantics
- $\Gamma \models_p \phi$ if $\Gamma \models_{Sta} \phi$: much trickier ! Relies on Van Fraassen (1976) who shows how, given an algebra of factual propositions and a probability function on it, to extend it with conditional propositions in such a way that (1) the conditional obeys Stalnaker C2 logic and (2) (AT) holds.

Summary and Perspectives

summary

- the Ramsey Test and its rendering in belief revision (RT_f); the logic induced
- Adams Thesis (probability of conditionals = conditional probability)
- Adams Logic equivalent on their common domain to Stalnaker Logic **C2**

Kaufmann on Adams Thesis

- to end up we will present you a scenario by S.Kaufmann (*JPL*, 2004) which is intended to show that intuitive judgements about probabilities of conditionals does not (always) obey (AT).
- a colored ball is picked out of one of two bags X and Y

$P(X) = 1/4$	$P(Y) = 3/4$
10 red balls	10 red balls
9 of them with a black spot	1 of them with a black spot
2 white balls	50 white balls

- $R \Rightarrow B$: “If I pick a red ball, it will have a black spot”
- Question: what is the probability of $R \Rightarrow B$?

Kaufmann on Adams Thesis, cont.

- most people answer “low”. Here is a possible reconstruction of their reasoning:

$$\begin{aligned} P(R \Rightarrow B) &= P(R \Rightarrow B|X)P(X) + P(R \Rightarrow B|Y)P(Y) \\ &\text{(Expansion by Case)} \\ &= P(B|RX)P(X) + P(B|RY)P(Y) \\ &\text{(Fact.Hypo)} \\ &= 9/10 \times 1/4 + 1/10 \times 3/4 = 0.3 \end{aligned}$$

- we’ll come back to Fact.Hypo according to which, in full generality, $P(A \Rightarrow C|B) = P(C|A \wedge B)$

Kaufmann on Adams Thesis, cont.

- but this is not conditional probability:

$$\begin{aligned} P(B|R) &= P(BR)/P(R) \\ &= P(B|RX)P(X|R) + P(B|RY)P(Y|R) \\ &= 9/10 \times 5/8 + 1/10 \times 3/8 = 0.6 \end{aligned}$$

- main difference: $P(B|RX)$ is multiplied by $P(X)$ in the intuitive computation and by $P(X|R)$ in the cond.prob. computation. In the second case, one takes into account that the fact a red ball has been picked changes the probabilities of X and Y.

Kaufmann on Adams Thesis, cont.

- Let's call *local probability of a conditional*(P_l) the probability calculated in the same way as our intuitive computation. In general, for a partition $X_1, ..., X_n$, $P_l(A \Rightarrow C) = P(C|AX_1) \cdot P(X_1) + ... + P(C|AX_n) \cdot P(X_n)$
- Kaufmann claims that
  (i) the probability of indicative conditionals goes sometimes by local proba. (and not conditional proba.), contrary to (AT)
  (ii) belief change goes by conditional proba. ((i) and (ii) contradicts (RT_p))
  (iii) it can be rational to evaluate the probability of indicative conditionals by local proba.

Kaufmann on Adams Thesis, cont.

- I disagree with (iii) : I am inclined to think that people are wrong when following “local probability”.
- Douven (forthcoming) shares this view and shows that local probability is inconsistent: the value of $P_l(A \Rightarrow C)$ is not invariant by the partition one considers.
- in Kaufmann’s scenario he divides urn X in two sub-urns X_1 and X_2 and shows that $P_l(R \Rightarrow B)$ calculated with X_1, X_2, Y differs from its value with X and Y

Kaufmann on Adams Thesis, cont.

- there is still something puzzling in the scenario - that is connected with the Triviality Results we will present tomorrow
- $P(.|X)$ is a probability function so by (AT) and laws of probability, it should be that $P(R \Rightarrow B|X) = P(B|RX)$
- but given Expansion by Case, it follows that $P(R \Rightarrow B) \neq P(B|R)$ which contradicts (AT) !!
- so it is not so clear that (AT) gives us a rational way of evaluating conditionals either !

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- Proof of $P_A(A \rightarrow C) \leq P(A \rightarrow C)$

$$P(A \rightarrow C|A) = P((\neg A \vee C) \wedge A)/P(A)$$
$$P(A \rightarrow C|A) = P(A \wedge C)/P(A) = P(C|A)$$

back

- another example of the failure of Transitivity by Bennett (2003). *A farmer believes strongly (but not with certainty) that the gate into the turnip field is closed and that his cows have not entered that field.*
- He believes strongly
  ✓ $A \Rightarrow B$: if the cows are in the turnip field, the gate has been left open
  ✓ $B \Rightarrow C$: if the gate to the turnip field has been left open, the cows have not noticed the gate's condition
- But he does not believe (strongly)
  ✓ $A \Rightarrow C$: if the cows are in the turnip field, they have not noticed the gate condition

back

degrees of belief in factual sentences

- basic idea: my degree of belief p_A in A is the **betting price** i.e. fair price that I assign to the bet
  ✓ 1 euro if A (net gain: 1 - p_A)
  ✓ 0 if $\neg A$ (net gain: - p_A)
- more generally: my degree of belief p_A is s.t. $p_A.S$ is the fair price that I assign to the bet
  ✓ S euro if A (net gain: $(1 - p_A).S$)
  ✓ 0 if $\neg A$ (net gain: $-p_A.S$)
- it turns out (**Dutch Book Theorem**) that if your set of betting prices violate the laws of probability, then there exists a set of bets that you should accept and that result in a sure loss

  back

example

- assume that for Paul $p_A + p_{\neg A} > 1$. Then the bookie can devise two bets:

✓ B1: *on A* with stake 1

✓ B2: *on* $\neg A$ with stake 1

an example, cont.

- here are the possible issues of the bets

w	B1	B2	Total Payoff
A	$(1 - p_A)$	$-p_{\neg A}$	$1 - p_A - p_{\neg A} < 0$
$\neg A$	$-p_A$	$(1 - p_{\neg A})$	$1 - p_A - p_{\neg A} < 0$

back

- what is my degree of belief in $A \Rightarrow C$?
- De Finetti (1937) introduces the notion of **conditional bet**
  ✓ S euro if AC (net gain: $(1-p).S$)
  ✓ 0 if $A\neg C$ (net gain: $-p.S$)
  ✓ called off if $\neg A$ (net gain: 0)
- does the fair price in a conditional bet reflects your degree of belief in $A \Rightarrow C$? If yes, then the **Conditional Dutch Book** shows that on pain of “incoherence”, $p_{A\Rightarrow C} = \frac{p_{AC}}{p_A}$!

conditional Dutch Book

- suspect that $p_{A \Rightarrow C} < \frac{p_{AC}}{p_A}$. Then the bookie can device three bets:

  ✓ B1: *on AC* with stake p_A

  ✓ B2: *against A* with stake p_{AC}

  ✓ B3: *against* $A \Rightarrow C$ with stake p_A

conditional Dutch Book, cont.

- here are the possible issues of the bets

w	B1	B2	B3
AC	$(1 - p_{AC}).p_A$	$-(1 - p_A)p_{AC}$	$-(1 - p_{A \Rightarrow C}).p_A$
$A\neg C$	$-p_{AC}.p_A$	$-(1 - p_A)p_{AC}$	$p_{\Rightarrow}.p_A$
$\neg A$	$-p_{AC}.p_A$	$p_{AC}.p_A$	0

- summing each line, one obtains as net results:

w	total payoff
AC	$p_{\Rightarrow}.p_A - p_{AC} < 0$
$A\neg C$	$p_{\Rightarrow}.p_A - p_{AC} < 0$
$\neg A$	0

back